ON COMMUTATIVE f-RINGS WHICH ARE RICH IN IDEMPOTENTS

By

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This work is lovingly dedicated to the memory of my father.

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In this dissertation we compare algebraic properties of commutative semi-prime f-rings A having the bounded inversion property with topological properties of $Max(A)$, the compact Hausdorff space of maximal ideals of A with the hull-kernel topology.

An f-ring A is local-global if for every primitive polynomial $f(t) \in A[t]$, there is an $a \in A$ such that $f(a)$ is a multiplicative unit. We prove that if A is a commutative semi-prime f-ring with identity and bounded inversion, then each of the following implies the next.

1. A is local-global.
2. For each primitive $a - bt^2 \in A[t]$ with $0 \leq a, b$, there exists a $c \in A$ such that $a - bc^2$ is a unit.
3. $Max(A)$ is zero-dimensional.

4. $Max(A) \cong Max(S(A))$, where $S(A)$ is a subalgebra of A generated by the idempotents of A.

Furthermore, the last three conditions are equivalent and if A has a strong unit, then all the conditions are equivalent. As a corollary, we show that for X a compact Hausdorff space that $C(X)$ is local-global if and only if X is zero-dimensional. As a special case we show that if A is a Bezout ring or X is a quasi-F space then the assumption of a strong unit or compactness can be dropped.

We then take a closer look at the containment of $S(A)$ in A. Specker-type and quasi-specker rings are defined. It is shown that A is a specker-type ring if and only if A is an f-subring of $Q(S(A))$, the complete ring of quotients of $S(A)$, if and only if $Q(A) = S(A)^L$, the lateral completion of $S(A)$.

We then define specker spaces and quasi-specker spaces. It is shown that a space X is a specker (quasi-specker) space if and only if βX, the Stone-Čech compactification, is a specker (quasi-specker) space. Finally we show that if X is a quasi-specker space and EX, the absolute of X, is a specker space, then X is a specker space. The converse obtains when X is compact and has a countable π-base.

CHAPTER 1
INTRODUCTION

This dissertation is largely an exploration of relationships that exist between certain algebraic structures and topological spaces that can be associated with one another in a natural and sometimes functorial manner.

The historical and methodological basis for this comparison goes back to the pioneering work of Marshall Stone, in particular his paper "Applications of the Theory of Boolean Rings to General Topology" [29] where he shows that the category of Boolean algebras with lattice homomorphisms is dual to the category of compact zero-dimensional Hausdorff spaces with continuous maps. This idea of considering certain algebraic substructures to be the points of a topological space pervades much of what follows.

1.1 Lattice Ordered Groups and the Yosida Space

A *lattice-ordered group*, denoted ℓ-group is a group $(G, +, 0)$ together with a partial ordering $\leq$ on G such that $(G, \leq)$ is a lattice satisfying a compatibility condition between the ordering and the group operation:

$$\text{If } a \leq b \text{ then } a + c \leq b + c \text{ and } c + a \leq c + b$$

We will explicitly define what a lattice is in Section 1.4. Although in the general theory of lattice-ordered groups there is no assumption or necessity that the group operation be commutative, for the purposes of this dissertation we will assume that all groups are commutative. A lattice-ordered group which is a vector space over the

reals such that scalar multiplication by positive real numbers preserves the order is called a *vector lattice.*

For $a, b \in G$ we will denote the least upper bound (join) and greatest lower bound (meet) of a and b by $a \vee b$ and $a \wedge b$ respectively. For infinite or arbitrary subsets of G we will denote respectively the least upper bound and greatest lower bound of these sets when they exist, by $\bigvee g_\lambda$ and $\bigwedge g_\lambda$ when they are indexed and $\bigvee S$ and $\bigwedge S$ otherwise. G^+ will denote the set of all $g \in G$ with $0 \leq g$.

Lattice-ordered groups have the following properties. A much more complete list of the properties of ℓ-groups as well as proofs of these results can be found in Chapter 1 of "Lattice-Ordered Groups" [2].

Let G be an ℓ-group, $a, b, c \in G$, then

1. $a + (b \vee c) = (a + b) \vee (a + c)$ and dually.

2. $-(a \vee b) = (-a) \wedge (-b)$ and dually.

3. $a + b = (a \vee b) + (a \wedge b)$.

4. $(G, \wedge, \vee)$ is a distributive lattice, that is $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and dually.

5. $(G, +)$ is a torsion free group.

Lattice-ordered groups also have the Riesz Interpolation Property;

6. If $h_1, \ldots, h_n \in G^+$ and if $0 \leq g \leq h_1 + \cdots + h_n$ then there exists $g_1, \ldots, g_n \in G^+$, with $0 \leq g_i \leq h_i$ for $1 \leq i \leq n$, such that $g = g_1 + \cdots + g_n$.

For $a, b \in G$ we say that a is *disjoint* to b if $a \wedge b = 0$. For $g \in G$ the *positive part* of g is $g^+ = g \vee 0$ and the *negative part* of g is $g^- = (-g) \vee 0$. Then $g^+ \wedge g^- = 0$ so that we can write any element uniquely as the difference of disjoint elements, namely

$g = g^+ - g^-$. The *absolute value* of g is $|g| = g^+ + g^-$. For (abelian) ℓ-groups we have the triangle inequality;

7. $|a+b| \leq |a| + |b|$.

From (3) above we have that;

8. If $a \wedge b = 0$ then $a + b = a \vee b$.

In the case that an infinite join or meet exists, we have the following infinite versions of (1), (2) and (4);

9. $g + (\bigvee h_\lambda) = \bigvee(g + h_\lambda)$ and dually.

10. $-(\bigvee h_\lambda) = \bigwedge(-h_\lambda)$ and dually.

11. $g \wedge (\bigvee h_\lambda) = \bigvee(g \wedge h_\lambda)$ and dually.

We now turn our attention to the subgroups of an ℓ-group.

Definition 1.1.1 *If G is an ℓ-group, a subgroup H of G is called an ℓ-subgroup if H is a sublattice of G; that is, H is closed under finite meets and joins. An ℓ-subgroup H is said to be* convex *if whenever $h_1 \leq g \leq h_2$ with $h_1, h_2 \in H$, then $g \in H$. A normal convex ℓ-subgroup is called an* ℓ-ideal. *If G and H are ℓ-groups, a map $\phi : G \rightarrow H$ is called an* ℓ-homomorphism *if it preserves both the group and lattice structure.*

Since we are dealing strictly with abelian ℓ-groups, normality of subgroups is not an issue. From now on we will not distinguish between convex ℓ-subgroups and ℓ-ideals. We have the following theorem relating ℓ-homomorphisms and ℓ-ideals.

Theorem 1.1.1 *(1.2.1 [2]) Let G and H be ℓ-groups and let $\phi : G \rightarrow H$ be an ℓ-homomorphism from G onto H.*

1. *$Ker\phi$, the kernel of ϕ, is an ℓ-ideal.*

2. *If N is an ℓ-ideal, then there is an ordering on G/N such that G/N is an ℓ-group and the canonical group homomorphism $\theta : G \to G/N$ is an ℓ-homomorphism.*

3. *$G/Ker\phi$ is ℓ-isomorphic to H.*

The ordering on G/N from (2) is induced by the ordering on G by defining $g+N \leq h+N$ if there is a $k \in N$ such that $g \leq h+k$. That this is a lattice ordering with $(g+N) \vee h+N = (g \vee h) + N$ and dually, is the content of Theorem 1.2.1 [2].

Let $\mathcal{C}(G)$ denote the set of ℓ-ideals of G. For $A, B \in \mathcal{C}(G)$, let $A \wedge B = A \cap B$ and $A \vee B =$ the subgroup generated by A and B. By convexity and the Riesz Interpolation Property, these operations make $\mathcal{C}(G)$ a lattice.

<u>Definition 1.1.2</u> *A lattice $\mathcal{A}$ is said to be complete if for every subset S of $\mathcal{A}$, both $\bigvee S$ and $\bigwedge S$ exist in $\mathcal{A}$. $\mathcal{A}$ is called brouwerian if for $a, b_\lambda \in \mathcal{A}$, $a \wedge (\bigvee b_\lambda) = \bigvee (a \wedge b_\lambda)$.*

We are now ready to state the following theorem which is a result from 1942 due to G. Birkhoff [6].

<u>Theorem 1.1.2</u> *$\mathcal{C}(G)$ is a complete brouwerian sublattice of the lattice of all subgroups of G.*

There are certain types of ℓ-ideals which are distinguishable in $\mathcal{C}(G)$ that will be of particular interest.

<u>Definition 1.1.3</u> *Let $g \in G$. $C \in \mathcal{C}(G)$ is said to be a* value *of g if C is maximal with respect to not containing g. C is said to be* prime *if G/C is totally ordered.*

We have the following theorems which allow us to distinguish values and prime ℓ-ideals in $\mathcal{C}(G)$. See Chapter 2 [5].

<u>Theorem 1.1.3</u> *Let* $C \in \mathcal{C}(G)$. *Then* $C \subsetneqq \bigcap\{B \in \mathcal{C}(G) : C \subsetneqq B\}$ *if and only if there is a* $g \in G$ *such that* C *is a value of* g.

An application of Zorn's Lemma establishes the following theorem.

<u>Theorem 1.1.4</u> *Let* $B \in \mathcal{C}(G)$ *with* $g \notin B$. *Then there is a value* C *of* g *with* $B \subset C$.

<u>Definition 1.1.4</u> *For a value* C, $C^* = \bigcap\{B \in \mathcal{C}(G) : B \subsetneqq C\}$ *is called the cover of* C.

The following provides several useful characterizations of prime ℓ-ideals.

<u>Theorem 1.1.5</u> *(Theorem 2.4.1 [5]) Let* $P \in \mathcal{C}(G)$. *The following are equivalent.*

1. P *is prime.*

2. *If* $A, B \in \mathcal{C}(G)$ *and* $A \cap B = P$ *then* $A = P$ *or* $B = P$.

3. $\{C \in \mathcal{C}(G) : P \subseteq C\}$ *is totally ordered.*

4. *For* $a, b \in G$, *if* $a \wedge b \in P$ *then* $a \in P$ *or* $b \in P$.

5. *For* $a, b \in G$, *if* $a \wedge b = 0$ *then* $a \in P$ *or* $b \in P$.

It is clear then that a value is prime, and from (4) above it appears that prime ℓ-ideals behave rather like prime ring ideals. We will eventually see in certain cases just how far this likeness goes. For now we have the following correspondence theorem.

<u>Theorem 1.1.6</u> *(Proposition 2.4.7 [5]) Let* G *be an* ℓ*-group,* H *an* ℓ*-subgroup of* G. *Then the map* $P \mapsto P \cap H$ *is a one-to-one correspondence between the prime* ℓ*-ideals of* G *not containing* H *and the proper prime* ℓ*-ideals of* H.

For a value V, $V^\star$ is an ℓ-group having V as an ℓ-subgroup. Since V is a prime ℓ-ideal of $V^\star$, by the correspondence theorem we get that the quotient group $V^\star/V$ is a totally ordered ℓ-group with no proper ℓ-ideals. The following will characterize a particular large class of totally ordered groups.

Definition 1.1.5 *An ℓ-group G is said to be archimedean if $na \leq b$ for all $n \in \mathbb{N}$ implies that $a \leq 0$.*

This is equivalent to, if $0 < a, b \in G$, then there is an $n \in \mathbb{N}$ such that $na \not\leq b$. Therefore in the totally ordered case this becomes the familiar definition of archimedeaneity.

The following theorem is due to O. Hölder [21]. It is from a paper published in 1901, and the proof is a generalization of the classical construction of the reals from the rationals using the cut completion.

Theorem 1.1.7 *Let G be a totally ordered group. The following are equivalent.*

1. *G is archimedean.*

2. *G is ℓ-isomorphic to an additive subgroup of $\mathbb{R}$.*

3. *G has no proper ℓ-ideals.*

We now turn our attention to one of the topological spaces which can be associated with an ℓ-group, namely the Yosida space of an archimedean ℓ-group.

Let $g \in G$ and denote by $G(g)$ the ℓ-ideal generated in G by g. That is $G(g) = \{a \in G : |a| \leq n|g| \text{ for some } n \in \mathbb{N}\}$. An element $0 < u \in G$ is called a *unit* if for any $a \in G$, if $a \wedge u = 0$ then $a = 0$; u is called a *strong unit* if $G(u) = G$. Note that a strong unit is a unit.

Suppose now that G is an archimedean ℓ-group with unit u. Let

$$Yos(G,u) = \{V \in \mathcal{C}(G) : V \text{ is a value of } u\}$$

The sets of the form $\mathcal{U}_a = \{V \in Yos(G,u) : a \notin V\}$, for all $a \in G$, are a base for the open sets of a topology on $Yos(G,u)$. We will call $Yos(G,u)$, with this topology the *Yosida space of G with respect to u*. This is in fact the hull-kernel topology on $Yos(G,u)$. We have the following theorem.

<u>Theorem 1.1.8</u> *(Corollary 10.2.5 [5]) Let G be an archimedean ℓ-group with unit u. Then $Yos(G,u)$ is a compact Hausdorff space.*

For a topological space X, denote by $D(X)$ the set of all continuous functions with values in the extended reals that are real valued on a dense subset of X. That is,

$$D(X) = \{f : X \to \mathbb{R} \cup \{\pm\infty\} : f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is dense in } X\}$$

It should be pointed out here that although $D(X)$ inherits a lattice ordering from the ordering on $\mathbb{R} \cup \{\pm\infty\}$, it is not generally true that $D(X)$ is a group. The problem is that one can define $f+g$ by $(f+g)(x) = f(x) + g(x)$ on $f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ and the domain of definition is a dense subset of X, but there is no guarantee that $f+g$ so defined can then be extended to all of X. If such an extension does exist, by the density of $f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ it is unique and in this case we define $f+g$ to be this unique extension. By an ℓ-group in $D(X)$ we mean a sublattice that is a subgroup with the group operation as defined above. The following is the Yosida Embedding Theorem for archimedean ℓ-groups.

<u>Theorem 1.1.9</u> *Let G be an archimedean ℓ-group with unit $u > 0$. Then there exists ℓ-embedding $\phi : G \to D(Yos(G,u))$ such that $\phi(G)$ is an ℓ-group and G is ℓ-isomorphic*

to $\phi(G)$. Moreover, ϕ can be taken so that $\phi(u) = \mathbf{1}$, the constant function 1, and if $V, W \in Yos(G,u)$ then there is a $g \in G$ such that $\phi(g)(V) \neq \phi(g)(W)$.

Theorem 1.1.9 originates in a 1942 paper by K. Yosida [34]. Hager and Robertson [19] show that the separation of points uniquely determines the Yosida space of an archimedean ℓ-group.

For later results we will need a precise description of the embedding ϕ whose existence is guaranteed by Theorem 1.1.9.

Let $V \in Yos(G,u)$. Since G is archimedean, by Hölder's theorem, $V^\star/V$ is ℓ-isomorphic to an additive subgroup of $\mathbb{R}$. Let θ_V be such an ℓ-isomorphism. Then $\phi : G \to D(Yos(G,u))$ is defined b, for $g \in G$,

$$\phi(g)(V) = \begin{cases} \theta_V(g+V) & \text{if } g \in V^\star \\ \infty & \text{if } g \notin V^\star \text{ and } g+V > V \\ -\infty & \text{if } g \notin V^\star \text{ and } g+V < V \end{cases}$$

1.2 Commutative Semi-Prime f-Rings and the Maximal Spectrum

For the purposes of this dissertation we will consider only commutative rings with identity. For more on f-rings, see chapter 9 in "Groupes et Anneaux Réticulés" [5].

Definition 1.2.1 *A lattice-ordered ring (ℓ-ring) is a ring $(A, +, \cdot, 0, 1)$ together with an ordering $\leq$ on the elements of A such that the following hold:*

1. *$(A, \leq, +, 0)$ is an ℓ-group.*

2. *For every $a, b, c \in A$, if $a \leq b$ and $0 \leq c$ then $ac \leq bc$ and $ca \leq cb$.*

Definition 1.2.2 *An ℓ-ring is said to be an f-ring if A is order isomorphic to a subdirect product of totally ordered rings.*

This is equivalent to,

3. For every $a, b, c \in A$ with $0 \leq c$, if $a \wedge b = 0$ then $ac \wedge b = 0 = ca \wedge b$.

By an *f-algebra* we mean an f-ring that is also a vector lattice, and by an *f-subring* we mean a subring that is also a sublattice. The following theorem lists some of the properties of f-rings.

Theorem 1.2.1 *(Theorem 9.1.10 [5]) Let A be an f-ring and let $a, b, c \in A$. Then,*

1. *If $c \geq 0$, then $c(a \wedge b) = ca \wedge cb$ and dually.*

2. *If $a \wedge b = 0$, then $ab = 0$.*

3. *$a^2 \geq 0$.*

4. *If $a \geq 0$ and $ab > 0$ then $b \geq 0$.*

5. *For $a, b \geq 0$, $ab = (a \wedge b)(a \vee b)$.*

If we are not careful, confusion can result in the context of lattice-ordered rings, due to the overlapping of terminology from ring theory and the theory of lattice-ordered groups. In an attempt to keep these ideas distinct and be consistent with acceptable usage, we repeat some old definitions and state some new ones. By an *ℓ-ideal* of A we mean a convex ℓ-subgroup of the ℓ-group $(A, +, \leq)$ and by an *ideal* of A we mean an ideal of the ring $(A, +, \cdot)$. An ideal P is a *prime ideal* if $ab \in P$ implies $a \in P$ or $b \in P$. An ℓ-ideal P is a *prime ℓ-ideal* if $a \wedge b \in P$ implies $a \in P$ or $b \in P$. An element $u \in A$ will be called a *multiplicative unit* if it is an unit in the ring $(A, +, \cdot)$, an *order unit* or *strong order unit* if it is such in the ℓ-group $(A, +, \leq)$. The following example is meant to help make these distinctions clear.

Example Let A be the set of all real sequences. If we define addition and multiplication pointwise, then A becomes a commutative ring. If we also order pointwise,

that is $\{a_n\} \geq \{b_n\}$ if $a_n \geq b_n$ for all $n \in \mathbb{N}$, then A is an f-ring. In fact if we also define real multiplication pointwise, then A is an f-algebra.

Let B be the set of bounded rational sequences. With operations and ordering defined as for A, B is a commutative f-ring. B is an f-subring of A, and B is not an f-algebra. Neither is B an ideal nor an ℓ-ideal of A.

Consider the sequences $a = \{1, 1/2, 1/3, \cdots\}$ and $b = \{1, 1, 1, \cdots\}$. In B, a is a weak order unit, but not a strong order unit or a multiplicative unit. In A, a is a weak order unit and a multiplicative unit, but not a strong order unit. In fact, A has no strong order units. As an element of B, b is a strong (hence weak) order unit and a multipicative unit. In A, b is a weak order unit and a multiplicative unit.

The following important concept is due to M. Henriksen et al.[20].

Definition 1.2.3 *An f-ring A is said to have the bounded inversion property if every $1 \leq a \in A$ is a multiplicative unit.*

Denote by $Spec(A)$, $Min(A)$, $Max(A)$ the set of prime ideals, minimal prime ideals and maximal ideals of A respectively. These become topological spaces when endowed with their respective hull-kernel topologies. In particular, the basic open sets of $Spec(A)$ are of the form, for $a \in A$, $\mathcal{M}_a = \{P \in Spec(A) : a \notin P\}$. A ring A is said to be *semi-prime* if it has no nonzero nilpotent elements. This is equivalent to requiring the intersection of prime ideals to be zero. The following results are well known and will be used extensively.

Lemma 1.2.1 *(Theorem 9.3.1 [5]) Let A be a semi-prime f-ring and let $a, b \in A$. Then $ab = 0$ if and only if $a \wedge b = 0$.*

Lemma 1.2.2 *(Theorem 9.3.1 [5]) Let A be a semi-prime f-ring. Then P is a minimal prime ideal if and only if P is a minimal prime ℓ-ideal.*

The following two results can be found in Henrikson et al.[20] where the notion of bounded inversion is introduced.

Lemma 1.2.3 *Let A be a semi-prime f-ring. Then A has the bounded inversion property if and only if every maximal ideal is an ℓ-ideal.*

Lemma 1.2.4 *Let A be a semi-prime f-ring with identity and bounded inversion. Then $Max(A)$ is a compact Hausdorff topological space.*

1.3 Tychonoff Spaces and Semi-Prime f-Rings

We will now look at the ring of continuous real valued functions. Let X be a topological space. Denote by $C(X)$ the set of all continuous real valued functions on X. For $f, g \in C(X)$, define

1. $f + g$ by $(f + g)(x) = f(x) + g(x)$ for all $x \in X$.

2. $f \cdot g$ by $(f \cdot g)(x) = f(x)g(x)$ for all $x \in X$.

We will write fg for $f \cdot g$. These operations make $(C(X), +, \cdot)$ a commutative ring. If in addition we define scalar multiplication by,

3. For $r \in \mathbb{R}$, rf by $(rf)(x) = r(f(x))$ for all $x \in X$.

Then $C(X)$ is a real vector space. We also define $\leq$, $\wedge$ and $\vee$ by,

4. $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$.

5. $f \wedge g$ by $(f \wedge g)(x) = min\{f(x), g(x)\}$, for each $x \in X$.

6. $f \vee g$ by $(f \vee g)(x) = max\{f(x), g(x)\}$, for each $x \in X$.

With these, $C(X)$ becomes a lattice ordered ring and a vector lattice. In addition, $C(X)$ is an f-ring and $C(X)$ has the bounded inversion property.

To distinguish topological spaces by the algebraic properties of $C(X)$, the next theorem will allow us to restrict our attention to the class of Tychonoff spaces. We first need the following definitions.

Definition 1.3.1 *Two subsets A and B of X are said to be completely separated if there is an $f \in C(X)$ such that $f(A) = 0$ and $f(B) = 1$.*

Definition 1.3.2 *A Hausdorff space X is said to be a Tychonoff space if for any closed set K and any $x \notin K$, $\{x\}$ and K are completely separated.*

Theorem 1.3.1 *(Theorem 3.9 [15]) For every topological space X, there exists a Tychonoff space Y and a continuous mapping τ from X onto Y such that the mapping $g \mapsto g \circ \tau$ is an isomorphism from $C(Y)$ onto $C(X)$*

This induced mapping is both a ring and a lattice isomorphism. For the remainder of this paper, unless explicitly stated otherwise, "space" will mean "Tychonoff space". For $Y \subset X$ we will denote the closure of Y in X by $cl_X(Y)$ or, when no ambiguity will result, by $cl(Y)$. Similarly we will denote the interior of Y in X by $int_X(Y)$ or $int(Y)$.

Definition 1.3.3 *For $f \in C(X)$ let $Z(f) = \{x \in X : f(x) = 0\}$. This is called the* zero set *of f. Let $coz(f) = \{x \in X : f(x) \neq 0\}$. This is called the cozero set of f. Let $Z(X) = \{Z(f) : f \in C(X)\}$.*

Tychonoff spaces have the following useful characterization.

Theorem 1.3.2 *(Theorem 3.2 [15]) A space X is a Tychonoff space if and only if $Z(X)$ is a base for the closed sets of X.*

It will occasionally be helpful to use this result as: X is Tychonoff if and only if the cozero sets are a base for the open sets of X.

What follows is a rudimentary description of the construction and properties of the Stone-Čech compactification of a Tychonoff space. A more detailed account can be found in Chapter 6 [15], or Chapter 1 [33].

Let X be a Tychonoff space and let $Z(X)$ denote the zero-sets of X. $\mathcal{F} \subset Z(X)$ is called a *z-filter* if,

1. $\emptyset \notin \mathcal{F}$.

2. If $Z_1, Z_2 \in \mathcal{F}$ then $Z_1 \cap Z_2 \in \mathcal{F}$.

3. If $Z \in \mathcal{F}$ and $Z \subset K \subset X$ then $K \in \mathcal{F}$.

A maximal z-filter is called a *z-ultrafilter*. Let βX denote the set of all z-ultrafilters of X. For $Z \in Z(X)$ let $\bar{Z} = \{\alpha \in \beta X : Z \in \alpha\}$. We topologize βX by taking $\{\bar{Z} : z \in Z(X)\}$ as a base for the closed sets. For $\alpha \in \beta X$, if $\cap\alpha \neq \emptyset$ then there is an $x \in X$ such that $\cap\alpha = \{x\}$. The map $x \mapsto \{\alpha \in \beta X : x \in \alpha\}$ is a dense embedding of X into βX. It is a theorem due to Gelfand and Kolmogoroff [13] that the map $p \mapsto \{f \in C(X) : p \in cl_{\beta X} Z(f)\}$ is a one to one correspondence between the points of βX and the maximal ideals of $C(X)$. In fact, this map is a homeomorphism from βX onto $Max(C(X))$. We will use this homeomorphism extensively.

Denote by $C^*(X)$ the set of all bounded continuous real valued functions on X. Then $C^*(X)$ is an f-subring of $C(X)$. We need to make the following definition.

<u>*Definition 1.3.4*</u> *Let $X \subset Y$ be topological spaces. X is said to be C^*-embedded in Y if every $f \in C^*(X)$ can be extended to a function in $C(Y)$.*

The following is most of Theorem 1.46 [33] and is a compilation of results characterizing βX.

<u>Theorem 1.3.3</u> *Every Tychonoff space X has a unique compactification βX which has the following equivalent properties:*

1. *X is C^*-embedded in βX.*

2. *Every continuous mapping of X into a compact space Y extends uniquely to a continuous map from βX into Y.*

3. *If Z_1 and Z_2 are zero sets in X, then $cl_{\beta X} Z_1 \cap cl_{\beta X} Z_2 = cl_{\beta X}(Z_1 \cap Z_2)$.*

4. *Disjoint zero sets in X have disjoint closures in βX.*

5. *Completely separated sets in X have disjoint closures in βX.*

6. *βX is the maximal compactification of X in the sense that if Y is any compactification of X, then there is a continuous map from βX onto Y that is the identity on X.*

Moreover, βX is unique in the sense that if T is a compactification of X satisfying any of the above conditions, then there is a homeomorphism of βX onto T that is the identity on X.

For the purposes of this dissertation, the major significance of Theorem 1.3.3 is that in many cases, we can without loss of generality assume the space we are dealing with is compact.

<u>1.4 Boolean Duality</u>

A *lattice* is a partially ordered set $(B, \leq)$ such that for every pair $\{a, b\} \subset B$, the least upper bound of a and b exists in B as does the greatest lower bound of a and b. We will denote the least upper bound of a and b by $a \vee b$ and call this the *join* of a and b, and we will denote the greatest lower bound of a and b by $a \wedge b$ and

call this the *meet* of a and b. A lattice B is said to be *bounded* if it has a least and largest element. In this section we will denote these, when they exist, by 0 and 1 respectively.

Definition 1.4.1 *A bounded lattice B is said to be complemented if for every $a \in B$ there is a $b \in B$ such that $a \wedge b = 0$ and $a \vee b = 1$. Such a b is called a complement of a.*

Definition 1.4.2 *A lattice B is said to be distributive if for every $a, b, c \in B$, $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.*

Definition 1.4.3 *A bounded distributive complemented lattice B is said to be a boolean algebra.*

In a boolean algebra, the complement of any element is unique and for $a \in B$ we will denote the complement of a by a'. For $a, b \in B$, we will denote the *difference* by $a \setminus b = a \wedge b'$. Boolean algebras have the following properties. A more complete list can be found in Chapter 1 of "Boolean Algebras" [28] or Chapter 3 in [27].

Let B be a boolean algebra and let $a, b, c \in B$. Then,

1. $(a \vee b)' = a' \wedge b'$ and dually.
2. $0' = 1$, $1' = 0$, $(a')' = a$.
3. $a \leq b$ if and only if $b' \leq a'$.
4. $a \wedge b = 0$ if and only if $a \leq b'$; $a \vee b$ if and only if $a' \leq b$.
5. For any finite $\{a_i : 1 \leq i \leq k\} \subset B$, $a \setminus (\bigvee a_i) = \bigwedge (a \setminus a_i)$.

Definition 1.4.4 *Let A, B be boolean algebras. A map $\theta : A \to B$ is called a boolean homomorphism if θ is a lattice homomorphism and for every $a \in A$, $\theta(a') = (\theta(a))'$.*

We are now going to look at the construction of the Stone space of a boolean algebra. This construction was originally carried out in M. Stone's 1936 paper, "Application of the theory of boolean rings to generel topology" [29]. Let B be a boolean algebra and let $\emptyset \neq \mathcal{F} \subset B$. $\mathcal{F}$ is called a *filter* on B or a *B-filter* if the following hold;

1. $0 \notin \mathcal{F}$.

2. If $a, b \in \mathcal{F}$ then $a \wedge b \in \mathcal{F}$.

3. If $a \in \mathcal{F}$ and $a \leq b \in B$, then $b \in \mathcal{F}$.

A B-filter $\mathcal{F}$ is called a *B-ultrafilter* if it is a maximal B-filter; that is, if $\mathcal{G}$ is a B-filter and $\mathcal{F} \subseteq \mathcal{G}$ then $\mathcal{F} = \mathcal{G}$. B-ultrafilters have the following characterization. See Section 2.3 [27].

Theorem 1.4.1 *Let $\mathcal{F}$ be a B-filter. The following are equivalent.*

1. *$\mathcal{F}$ is a B-ultrafilter.*

2. *For every $a \in B$, if $a \wedge b \neq 0$ for every $b \in \mathcal{F}$ then $a \in \mathcal{F}$.*

3. *For every $a \in B$, either $a \in \mathcal{F}$ or $a' \in \mathcal{F}$.*

For a boolean algebra B, let $St(B)$ denote the set of B-ultrafilters. For $b \in B$ let $\mathcal{U}_b = \{\mathcal{F} \in St(B) : b \notin \mathcal{F}\}$. We topologize $St(B)$ by taking $\{\mathcal{U}_b : b \in B\}$ as a base for the open sets. Since we will see this kind of construction again, we will look closely here at the relationship between intersection, union and complementation of basic open sets in $St(B)$, and meet, join and complementation of elements of B.

Let $a, b \in B$. The following can be easily verified.

1. $\mathcal{U}_a \cap \mathcal{U}_b = \mathcal{U}_{a \vee b}$.

2. $\mathcal{U}_a \cup \mathcal{U}_b = \mathcal{U}_{a \wedge b}$.

3. $St(B) \setminus \mathcal{U}_a = \mathcal{U}_{a'}$.

4. $\mathcal{U}_1 = \emptyset$ and $\mathcal{U}_0 = St(B)$.

If $\mathcal{F} \in St(B)$, since $\mathcal{F}$ is a B-ultrafilter, for $0 \neq b \in B$ either $b \in \mathcal{F}$ or $b' \in \mathcal{F}$. Therefore either $\mathcal{F} \in \mathcal{U}_{b'}$ or $\mathcal{F} \in \mathcal{U}_b$. If $\mathcal{F} \in \mathcal{U}_a \cap \mathcal{U}_b$ then $a \vee b \notin \mathcal{F}$ so that $\mathcal{F} \in \mathcal{U}_{a \vee b}$. By (1) above, $\{\mathcal{U}_b : b \in B\}$ is a base for the open sets of a topology on $St(B)$.

Let X be a Hausdorff space. A subset K of X is called *clopen* if K is both closed and open. Let $\mathcal{B}(X)$ denote the set of clopen sets of X. With the operations of set theoretic union ($\cup$), intersection ($\cap$) and complementation ($'$), $(\mathcal{B}(X), \cup, \cap, ')$, is a boolean algebra.

Definition 1.4.5 *A Hausdorff space X is said to be zero-dimensional if $\mathcal{B}(X)$ is a base for the open sets of X.*

Definition 1.4.6 *A Hausdorff space X is said to be totally disconnected if for every $x \in X$, the connected component of x is $\{x\}$.*

We will make use of the following result which is Theorem 16.17 in [15].

Theorem 1.4.2 *Let X be a Tychonoff space. Each of the following conditions implies the next and if X is compact, all conditions are equivalent.*

1. *Disjoint zero sets are contained in disjoint clopen sets.*

2. *X is zero-dimensional.*

3. *X is totally disconnected.*

We are now ready to state Stone's Representation Theorem. The following version is Theorem 3.2 (d) [27].

Theorem 1.4.3 *Let B be a boolean algebra. Then,*

1. *$St(B)$ is a compact zero-dimensional Hausdorff space.*

2. *$\{\mathcal{U}_b : b \in B\} = \mathcal{B}(St(B))$.*

3. *The map $b \mapsto \mathcal{U}_b$ is a boolean isomorphism from B onto $\mathcal{B}(St(B))$.*

Categorically Let **CZD** denote the category having as objects compact zero-dimensional Hausdorff spaces and continuous maps as morphisms. Let **BA** be the category of boolean algebras and boolean maps.

Define F : **CZD** $\rightarrow$ **BA** by $F(X) = \mathcal{B}(X)$ for objects and if $f : X \rightarrow Y$ is a morphism, define $F(f) : \mathcal{B}(Y) \rightarrow \mathcal{B}(X)$ by: for $K \in \mathcal{B}(Y)$, $F(f)(K) = f^{-1}(K)$. It can then be shown that F is a contravariant functor from **CZD** to **BA**.

Define G : **BA** $\rightarrow$ **CZD** by $G(B) = St(B)$ for objects and if $\theta : A \rightarrow B$ is a morphism, define $G(\theta) : St(B) \rightarrow St(A)$ by: for $\mathcal{F} \in St(B)$, $G(\theta)(\mathcal{F}) = \theta^{-1}(\mathcal{F})$. G is a contravariant functor from **BA** to **CZD**. Moreover, as (3) from Stone's Theorem may indicate, **CZD** and **BA** are dual categories.

CHAPTER 2
LOCAL-GLOBAL f-RINGS

2.1 Introduction

Let A be a commutative semi-prime f-ring with identity and bounded inversion. We first need to recall some ideas from commutative ring theory (see Chapters 2 and 4 in "Multiplicative Ideal Theory"[16]).

A subset S of A is called a *regular multiplicative closed subset* if $0 \notin S$, S is closed under multiplication and S contains no divisors of zero. Denote by $S^{-1}A$ the set of all fractions a/s where $a \in A$ and $s \in S$. If we define equality, addition and multiplication as in the classical construction of the quotient field of an integral domain, then $S^{-1}A$ is a commutative semi-prime ring with identity and the map $a \mapsto (as)/s$ is a monomorphism from A into $S^{-1}A$. If $1 \in S$, we can take $s = 1$ and this map becomes $a \mapsto a/1$. $S^{-1}A$ is called the *ring of quotients of A with respect to S*. If we take S to be the set of all regular elements of A, then $S^{-1}A$ is called the *classical ring of quotients* of A and will be denoted by qA. We can define an ordering on qA by first observing that for a/s we can assume that $s > 0$, since $a/s = as/s^2$. Then define $(a/s) \vee 0 = (a \vee 0)/s$. This extends the ordering on A in such a way that qA is an f-ring and the embedding of A into qA preserves order. If $a/s > 1$ in qA, then $a > s$ and since s is regular so is a. Therefore, qA has bounded inversion.

We now look at the *localization* of A at a prime ideal P. Let $O(P) = \{a \in A : ab = 0 \text{ for some } b \notin P\}$. Then it can be shown that $O(P)$ is the intersection of the minimal prime ℓ-ideals contained in P and therefore that $O(P)$ is an ideal and an ℓ-ideal. We define A_P to be the ring of quotients of $A/O(P)$ with respect

to $S = \{b + O(P) : b \notin P\}$. The point of factoring out $O(P)$ is to insure that S contains no divisors of zero. In the case where M is a maximal ideal, since A is an f-ring with bounded inversion, $A/O(M)$ is already a local ring having $M/O(M)$ as its unique maximal ideal. Therefore the elements of S are already invertible and so $A_M \cong A/O(M)$. In the context of commutative semi-prime f-rings with bounded inversion, when we refer to the localization at a maximal ideal we will mean the quotient ring $A/O(M)$.

We need the following definitions.

Definition 2.1.1 *A polynomial $f \in A[x_1, ..., x_n]$ is said to represent a unit over A if there exists $a_1, ..., a_n \in A$ such that $f(a_1, ..., a_n)$ is a multiplicative unit in A.*

For $f \in A[x_1, ..., x_n]$, let f_M denote the polynomial in $A_M[x_1, ..., x_n]$ whose coefficients are the respective images of the coefficients of f under the map $a \mapsto a/1$.

Definition 2.1.2 *A ring A is said to have the local-global property if for each polynomial $f \in A[x_1, ..., x_n]$, whenever f_M represents a unit over A_M, for all $M \in Max(A)$, then f represents a unit over A.*

In their paper "Module Equivalences" [11], Estes and Guralnick look at many of the implications of the local-global property. The authors first point out that historically, many of the results obtained for local-global rings are extensions of results first obtained by R. S. Pierce for rings which are Von Neumann regular modulo their Jacobson radical [26]. In particular, for modules over local-global rings, $M_1 \oplus N \simeq M_2 \oplus N$ implies $M_1 \simeq M_2$, and $M^n \simeq N^n$ implies $M \simeq N$.

Recall that a polynomial $f(x) \in A[x]$ having coefficients $a_1, \ldots, a_n$ is said to be *primitive* if there exists $b_1, \ldots, b_n \in A$ such that $a_1 b_1 + \cdots + a_n b_n = 1$.

<u>*Definition 2.1.3*</u> *A ring A is said to satisfy the primitive criterion if for any primitive polynomial $f \in A[x]$, f represents a unit over A.*

By a *residue field* we mean the field of quotients of the integral domain A/P for P a prime ideal.

The following result, due to Estes and Guralnick [11], will allow us to determine when an f-ring is local-global in terms of the primitive criterion.

<u>*Lemma 2.1.1*</u> *A ring A satisfies the primitive criterion if and only if A is a local-global ring with all residue fields infinite.*

Consequently, if A is a commutative f-ring then A is local-global if and only if A satisfies the primitive criterion. We will use this characterization of local-global f-rings to eventually characterize the local-global property in terms of $Max(A)$.

Recall that for an f-ring A, $B \subset A$ is said to be an *f-subring* if B is a subring and a sublattice of A. We have the following theorem relating an f-subring of an f-ring and their respective maximal spectra.

<u>*Theorem 2.1.1*</u> *Let A be a commutative semi-prime f-ring with identity and bounded inversion. If B is an f-subring of A with bounded inversion and $1 \in B$ then the map*

$$\phi : Max(A) \rightarrow Max(B)$$

defined by $\phi(M) =$ the unique maximal ideal containing $M \cap B$, is a continous surjection.

PROOF

We first note that for $M \in Max(A)$, $M \cap B$ is a prime ideal and a prime ℓ-ideal of B. Therefore $M \cap B \subset M' \in Max(B)$ for some M' and as this is also a prime

ℓ-ideal and the prime ℓ-ideals form a root system, M' is unique. Therefore, ϕ is well defined.

To show that ϕ is onto, let $N \in Max(B)$. Since $1 \notin N$, there is a value V of 1 in B with $N \subset V$. Let W be the ℓ-ideal generated by V in A; that is $W = \{a \in A : |a| \leq v$ for some $v \in V\}$. Then $1 \notin W$ so that $W \subset W'$, a value of 1 in A. Let P be a minimal prime ℓ-ideal contained in W'. Then P is a minimal prime ideal, so that $P \subset M$ for some $M \in Max(A)$. Since M is a prime ℓ-ideal, $P \subset M \subset W'$. Now, $W' \cap B$ is a prime ℓ-ideal of B containing V with $1 \notin W' \cap B$. As V is a value of 1, $V = W' \cap B$ and $V \supset M \cap B$. Since $M \cap B$ is a prime ideal and a prime ℓ-ideal of B it is contained in a unique maximal ideal $\phi(M)$ of B with $\phi(M) \subset V$. If $N \neq \phi(M)$, then $B = N + \phi(M) \subset N \vee \phi(M) \subset V$. This is a contradiction. Therefore, $\phi(M) = N$, so that ϕ is onto.

To show that ϕ is continous, we will consider ϕ as the composite $\mu \circ \rho$ where $\mu : Spec(B) \rightarrow Max(B)$ is defined by $\mu(P) =$ the unique maximal ideal containing P and $\rho : Max(A) \rightarrow Spec(B)$ is defined by $\rho(M) = M \cap B$. To see that ρ is continous, let $x \in B$ and let $\mathcal{N}_x = \{P \in Spec(B) : x \notin P\}$. This is a basic open set for the hull-kernel topology on $Spec(B)$. Since $x \in B \subset A$, $\mathcal{M}_x = \{M \in Max(A) : x \notin A\}$ is a basic open set for the hull-kernel topology on $Max(A)$. It is easy to see that $M \in \mathcal{M}_x$ if and only if $M \cap B \in \mathcal{N}_x$, so that $\rho^{-1}(\mathcal{N}_x) = \mathcal{M}_x$ and ρ is continous. To show that μ is continuous, we will show continuity at an arbitrary point. This part of the proof is modeled on Lemma 10.2.3 [5]. Let $P \in Spec(B)$ and suppose that U is an open neighborhood of $\mu(P)$ in $Max(B)$. Without loss of generality, we may assume that U is a basic open set in $Max(B)$. Say $U = \mathcal{M}_x = \{M \in Max(B) : x \notin M\}$. Then $U = Max(B) \cap \mathcal{N}_x$, where $\mathcal{N}_x = \{P \in Spec(B) : x \notin P\}$ is a basic open set in $Spec(B)$. Let $Q \in Max(B) \setminus \mathcal{N}_x$. Then Q and $\mu(P)$ are distinct maximal ideals. Since Q and $\mu(P)$ are also ℓ-ideals, we can find $0 \leq x \in Q \setminus \mu(P)$ and $0 \leq y \in \mu(P) \setminus Q$ with $x \wedge y = 0$,

so that Q and $\mu(P)$ are contained in disjoint open sets in $Spec(B)$. Suppose $Q \in U_Q$ and $\mu(P) \in V_Q$, both open with $U_Q \cap V_Q = \emptyset$. Consider $\{U_Q : Q \in Max(B) \setminus \mathcal{N}_x\}$, where the U_Q are as above. Since any neighborhood of a maximal ideal M in $Spec(B)$ is a neighborhood of any prime $P \subset M$, $\{U_Q : Q \in Max(B) \setminus \mathcal{N}_x\}$ is an open cover of $Spec(B) \setminus \mathcal{N}_x$. Now, $Spec(B) \setminus \mathcal{N}_x$ is a closed subspace of of $Spec(B)$ and so is compact. Therefore there exists $\{Q_i, 1 \leq i \leq n\}$, such that $Spec(B) \setminus \mathcal{N}_x \subset \bigcup_{i=1}^{n} U_{Q_i}$. Let $V = \bigcap_{i=1}^{n} V_{Q_i}$, where the V_{Q_i} are as above. Then $\mu(P) \in V$ so that $P \in V$ and $V \subset \mathcal{N}_x$. Then $\mu(V) \subset \mu(\mathcal{N}_x) = \mathcal{M}_x$ and therefore, μ is continuous.

We have shown that $\phi = \mu \circ \rho$ and that this is a continuous surjection from $Max(A)$ to $Max(B)$.

QED

Of particular interest in what follows will be the occasions when the map ϕ defined in the preceeding theorem is one-to-one. To describe these occurences we make the following definition.

Definition 2.1.4 *Let A and B be as in the statement of Theorem 2.1.1, and let ϕ be the continuous surjection guaranteed by the theorem. If ϕ is a homeomorphism, we say that B separates the points of A.*

2.2 The Specker Subring of a Semi-Prime f-Ring

For A an f-ring, let $A(1)$ denote the set of bounded elements of A. That is $A(1) = \{a \in A : |a| \leq n \cdot 1 \text{ for some } n \in \mathbb{N}\}$. Then $A(1)$ is an f-subring and an ℓ-ideal of A. If $A = A(1)$ we say that A is *bounded*. We have the following corollary to Theorem 2.1.1.

Corollary 2.2.1 *Let A be a commutative semi-prime f-ring with identity and bounded inversion. Then $Max(A) \cong Max(A(1))$.*

PROOF

We first note that if A has bounded inversion then so does $A(1)$, so that both $Max(A)$ and $Max(A(1))$ are compact Hausdorff. Since $A(1)$ is an f-subring of A with $1 \in A(1)$, by Theorem 2.1.1, it suffices to show that the map $\phi : Max(A) \to Max(A(1))$ is one-to-one. Recall that for $M \in Max(A)$, $\phi(M)$ is the unique element of $Max(A(1))$ containing $M \cap A(1)$. Suppose that $N, M \in Max(A)$ with $N \neq M$. If $\phi(N) = \phi(M)$, since $A(1) \in \mathcal{C}(A)$ and $\mathcal{C}(A)$ is distributive, we have that

$$\phi(M) \supset (N \cap A(1)) \vee (M \cap A(1)) = (N \vee M) \cap A(1) = (N + M) \cap A(1) = A \cap A(1) = A(1)$$

This is a contradiction. Therefore, $\phi(N) \neq \phi(M)$ so that ϕ is one-to-one.

QED

For A a semi-prime f-ring with bounded inversion, if $n \in \mathbb{N}$, then $n \cdot 1 \geq 1$ in A so that $(n \cdot 1)^{-1} \in A$. In particular, A is divisible, so that we may consider A as an algebra over $\mathbb{Q}$. Let $S(A)$ denote the $\mathbb{Q}$ subalgebra generated by the idempotents of A. We will call this the *specker subring* of A. If A is a vector lattice, the specker subring of A is $S_{\mathbb{R}}(A)$ the real subalgebra generated by the idempotents of A.

Definition 2.2.1 *An ℓ-group G is called hyper-archimedean if every ℓ-homomorphic image is archimedean.*

We will call an f-ring A hyper-archimedean if its ℓ-group structure is hyper-archimedean. Most of following theorem was obtained originally for vector lattices, and is due in this more general form to P. Conrad [8].

Theorem 2.2.1 Let G be an ℓ-group. The following are equivalent.

1. G is hyper-archimedean.

2. Each proper prime ℓ-ideal is maximal and hence minimal.

3. $G = G(g) \boxplus g^{\perp}$ for each $g \in G$.

4. G is ℓ-isomorphic to an ℓ-subgroup G^* of $\prod \mathbb{R}_i$ and for each $0 < x, y \in G^*$ there exists an $n > 0$ such that $nx_i > y_i$ for all $x_i \neq 0$.

5. If $0 < a, b \in G$ then $a \wedge nb = a \wedge (n+1)b$ for some $n > 0$.

We also have the following characterization which will be useful in the present context.

Theorem 2.2.2 (Theorem 14.1.7 [5]) An ℓ-group G is hyper-archimedean if and only if G is ℓ-isomorphic to an ℓ-subgroup of $C(X)$ such that G separates the points of X and the support of each $g \in G$ is compact and open.

Since $S(A)$ is archimedean and has a strong order unit, by Yosida embedding, $S(A)$ embeds as an ℓ-subgroup of $C(Yos(S(A)), 1))$ and $S(A)$ separates the points of $Yos(S(A))$. Since each $a \in S(A)$ is a finite sum of idempotents and idempotents must map to the characteristic functions of clopen sets, the support of a is clopen hence compact and open. We then have most of the following lemma.

Lemma 2.2.1 For A a commutative semi-prime f-ring with identity and bounded inversion, $S(A)$ is hyper-archimedean. If A is also a vector lattice then $S_{\mathbb{R}}(A)$ is hyper-archimedean.

Theorem 2.2.3 Let A be a commutative semi-prime f-ring with identity and bounded inversion. If $S(A)$ is convex, then A is hyper-archimedean and $A = S(A)$.

PROOF

Suppose that $S(A)$ is convex in A. By Theorem 1.1.6, the map $P \mapsto P \cap S(A)$ is a one to one order preserving correspondence between the prime ℓ-ideals of A not containing $S(A)$ and the proper prime ℓ-ideals of $S(A)$. Since $S(A)$ is hyper-archimedean, the prime ℓ-ideals of $S(A)$ are trivially ordered and consequently, so are the prime ℓ-ideals of A which do not contain $S(A)$. We will show that no prime ℓ-ideal of A contains $S(A)$.

Let P be a prime ℓ-ideal of A. Then $P \subset V$ a value of 1. $V \supset Q$ a minimal prime ℓ-ideal and since A is semi-prime, Q is also a prime ideal and so is contained in a maximal ideal M. Since A has bounded inversion, M is an prime ℓ-ideal and so, $Q \subset M \subset V$ and $Q \subset P \subset V$. We will show that $P \subset M$. Now consider the quotient ring A/M. Since M is both a maximal ideal a prime ℓ-ideal, A/M is an ordered field. Since $S(A)$ and M are convex, $(S(A)+M)/M$ is a convex f-subring of A/M. Also, as $1 \notin M$, $M \not\supseteq S(A)$ so that $M \cap S(A)$ is a proper prime ℓ-ideal of $S(A)$. Since $S(A)$ is hyper-archimedean, $S(A)/(S(A) \cap M)$ is archimedean. By Theorem 2.3.9 [5], we have that $S(A)/(S(A) \cap M) \cong (S(A) + M)/M$ as ℓ-groups and hence as f-rings. Suppose now that $x \geq 1$ in A/M. Then $x^{-1} \ll 1$, but $1 \in (S(A)+M)/M$ and this is convex and archimedean; a contradiction. Therefore, A/M is a real field, in particular $V/M = 0$ so that $V = M$. Then $P \subset M$ and since $M \not\supseteq S(A)$, we have that $P \not\supseteq S(A)$. We have shown that any prime ℓ-ideal does not contain $S(A)$ and therefore the prime ℓ-ideals of A are trivially ordered. By Theorem 2.2.1, A is hyper-archimedean. Clearly, $S(A) \subset A(1)$. Since $1 \in S(A)$ and $S(A)$ is convex, $A(1) \subset S(A)$. Since A is hyper-archimedean with identity, by Lemma A [8], $A = A(1)$. Therefore $A = S(A)$.

QED

We should note that if A is, in addition, a vector lattice, by the above and Proposition 1.2 [8], the following are equivalent.

1. A is hyper-archimedean.

2. $A = S_{\mathbb{R}}(A)$.

3. $S_{\mathbb{R}}(A)$ is convex in A.

We will also need the following lemma.

<u>*Lemma 2.2.2*</u> *If A is a commutative semi-prime f-ring with identity and bounded inversion, then so is $S(A)$. If A is a vector lattice this holds for $S_{\mathbb{R}}(A)$.*

PROOF

That $S(A)$ is a commutative semi-prime f-ring with identity follows from $S(A)$ being an f-subring of A and $1^2 = 1$. We need only show that $S(A)$ has bounded inversion. Let $a \in S(A)$. Then $a = \sum_{i=1}^{n} q_i e_i$ where for $1 \leq i \leq n$, $q_i \in \mathbb{Q}$ and $e_i \in A$ is idempotent. We first note that a can be written as $a = \sum_{i=1}^{m} r_i f_i$ where the $r_i \in \mathbb{Q}$ are distinct and more importantly, the $f_i \in A$ are idempotents such that $f_i f_j = 0$ for $i \neq j$. Suppose now that $1 \leq a \in S(A)$. Then there is a $b \in A$ such that $ba = 1$. That is $b(\sum_{i=1}^{m} r_i f_i) = \sum_{i=1}^{m} b r_i f_i = 1$. For each $1 \leq i \leq m$ we get that $b r_i f_i = f_i$, so that $\sum_{i=1}^{m} f_i = 1$. Let $c = \sum_{i=1}^{m} (\frac{1}{r_i}) f_i$. Then $c \in S(A)$ and $ac = \sum_{i=1}^{m} f_i = 1$. Therefore, $S(A)$ has bounded inversion. The proof is similar for $S_{\mathbb{R}}(A)$, taking real coefficients instead of rational.

QED

2.3 A Characterization of Local-Global f-Rings

The main result of this section is the following theorem.

Theorem 2.3.1 *Let A be a commutative semi-prime f-ring with identity and bounded inversion. Each of the following conditions implies the next one.*

1. *A is local-global.*
2. *If $a - bt^2 \in A[t]$ is primitive with positive coefficients, then $a - bt^2$ represents a multiplicative unit in A.*
3. *$Max(A)$ is zero-dimensional.*
4. *$Max(A) \cong Max(S(A))$. (If A is a vector lattice, $Max(A) \cong Max(S_{\mathbb{R}}(A))$.)*

Furthermore, (2), (3) and (4) are always equivalent and if A is bounded, then all four conditions are equivalent.

We proceed with a series of lemmas which will be used in the proof of this theorem.

Lemma 2.3.1 *(Proposition 2.1 [9]) $Min(A)$ is a zero dimensional Hausdorff space.*

For a ring A, let $J(A)$ denote the Jacobson radical of A. Then $J(A) = \bigcap\{M : M \in Max(A)\}$. We have the following lemma.

Lemma 2.3.2 *Let A be a commutative semi-prime f-ring with identity and bounded inversion. Then $Max(A) \cong Max(A/J(A))$.*

PROOF

We first observe that since $Max(A) \subset \mathcal{C}(A)$, $J(A)$ is an ℓ-ideal of A so that the canonical map $A \to A/J(A)$ is an ℓ-homomorphism as well as a ring homomorphism. Consequently, $A/J(A)$ is a commutative semi-prime f-ring with identity and bounded inversion. The canonical map $A \mapsto A/J(A)$ induces a one-to-one correspondence between the maximal ideals of A and the maximal ideals of $A/J(A)$. Let $\theta : Max(A) \to Max(A/J(A))$ be this induced map. Since $Max(A)$ and $Max(A/J(A))$ are both compact Hausdorff spaces, we need only show that θ is continuous. Let $\mathcal{M}_x = \{M \in Max(A) : x \notin M\}$ be a basic open set of $Max(A)$. Since $M \supset J(A)$ for every $M \in Max(A)$, we have that

$$M \in \mathcal{M}_x \Leftrightarrow x \notin M \Leftrightarrow \overline{x} \notin \theta(M) \Leftrightarrow \theta(M) \in \mathcal{M}_{\overline{x}},$$

where $x \in A$ and $\overline{x} \in Max(A/J)$ are such that $x \mapsto \overline{x}$ under the canonical map, and $\mathcal{M}_{\overline{x}} = \{M \in Max(A/J) : \overline{x} \notin M\}$. Thus $\theta^{-1}(\mathcal{M}_{\overline{x}}) = \mathcal{M}_x$. Therefore, θ is continuous.

QED

We have seen that the maximal spectra of A and $A/J(A)$ are homeomorphic. We now consider some algebraic properties of A and $A/J(A)$.

Lemma 2.3.3 *Let A be a commutative semi-prime f-ring with identity and bounded inversion. Then A is local-global if and only if $A/J(A)$ is local-global.*

PROOF

Let $p(t) = a_0 + a_1t + \cdots + a_nt^n \in A[t]$ and let $\overline{p}(t) = \overline{a}_0 + \overline{a}_1t + \cdots + \overline{a}_nt^n \in (A/J(A))[t]$ where a_i and $\overline{a}_i$ are such that $a_i \mapsto \overline{a}_i$ under the canonical map $A \to A/J$. By Lemma 2.1.1, we need only show that $p(t)$ is primitive if and only if $\overline{p}(t)$ is. It is clear that if $p(t)$ is primitive, then so is $\overline{p}(t)$. Suppose now that $\overline{p}(t)$ is primitive.

Then there exists $\overline{b}_0, \overline{b}_1, \cdots, \overline{b}_n \in A/J(A)$ such that $a_0b_0 + a_1b_1 + \cdots + a_nb_n = 1 + j$ for some $j \in J(A)$. If $a_0b_0 + a_1b_1 + \cdots + a_nb_n = 1 + j \in M$ for some $M \in Max(A)$, then $1 = a_0b_0 + a_1b_1 + \cdots + a_nb_n - j \in M$ as $j \in J(A) \subset M$. This is a contradiction. Therefore, $a_0b_0 + a_1b_1 + \cdots + a_nb_n$ is a unit, so that $p(t)$ is primitive and the result follows.

QED

Lemma 2.3.4 *Let A be a commutative semi-prime f-ring with identity and bounded inversion. Then $Yos(A, 1) \cong Max(A)$. If in addition A is bounded then $Yos(A, 1) = Max(A)$.*

PROOF

For each $M \in Max(A)$, M is a prime ℓ-ideal with $1 \notin M$ and so it is contained in a unique value $V \in Yos(A, 1)$. For $V \in Yos(A, 1)$, $V \supseteq P$ a minimal prime ℓ-ideal and a minimal prime ideal. Then there is an $M \in Max(A)$ with $P \subset M \subset V$. If $M_1 \neq M_2 \in Max(A)$ with $M_1, M_2 \subset V$ then $A = M_1 + M_2 \subset M_1 \vee M_2 \subset V$, so that each value of 1 in A contains exactly one maximal ideal. As in the proof of Theorem 2.1.1, this correspondence induces the required homeomorphism.

Suppose that A is bounded. Since the topologies on both spaces are their respective hull-kernel topologies, it suffices to show that $Yos(A, 1) = Max(A)$ as sets. Let $Y \in Yos(A, 1)$. $Y \supset P$ a minimal prime ℓ-ideal, hence a minimal prime ideal. Then $P \subset M \in Max(A)$ with $P \subset M \subset Y$. Now consider the quotient ring A/M. Since M is a maximal ideal, A/M is a field. Since M is also a prime ℓ-ideal, A/M is an ordered field. Since A is bounded, A/M is as well. Therefore, A/M is order-isomorphic to a subfield of $\mathbb{R}$. Now, Y/M is an ℓ-ideal of A/M and so by Hölder's Theorem,

$Y/M = 0$. Therefore, $Y = M$, so that Y is a maximal ideal of A. We then have that $Yos(A,1) \subset Max(A)$. By the above homeomorphism, $Yos(A,1) = Max(A)$.

QED

For the next result we will need the following definition.

Definition 2.3.1 Let G be an ℓ-group. For $0 \leq a, b \in G$ we say that a is infinitesimal to b, denoted $a \ll b$, if $na \leq b$ for all $n \in \mathbb{N}$.

Lemma 2.3.5 Let A be a bounded commutative semi-prime f-ring with identity and bounded inversion. Then $J(A) = \{x \in A : |x| \ll 1\}$ and A is archimedean if and only if $J(A) = 0$.

PROOF

Let $x \in J(A)$. Since $J(A)$ is convex, we may assume that $x \geq 0$. Suppose now that for some $n \in \mathbb{N}$, $nx \nleq 1$. Then $(nx-1)^+ > 0$, so there is a minimal prime ℓ-ideal P with $(nx-1)^+ > 0 \ mod\ P$. Since P is prime, $(nx-1)^- = 0 \ mod\ P$, so that $(nx-1) > 0 \ mod\ P$ and $nx > 1 \ mod\ P$. Now, P is a minimal prime ideal so there is an $M \in Max(A)$ with $P \subset M$. The canonical map $a + P \mapsto a + M$ is a lattice homomorphism from A/P to A/M so that $nx > 1 \ mod\ M$. But, $x \in J(A) \subset M$ and M is convex so that $1 \in M$; a contradiction. Therefore, $x \ll 1$.

Suppose now that $|x| \ll 1$. Since each $M \in Max(A)$ is a prime ℓ-ideal, $n|x| \leq 1 \ mod\ M$ for all $M \in Max(A)$, $n \in \mathbb{N}$. By Lemma 2.3.4, each $M \in Max(A)$ is a value of 1 in A. Since A is bounded, $A/M = M^*/M$ is archimedean so that $|x| = 0 \ mod\ M$, whence $x \in M$. Since this holds for each $M \in Max(A)$, we have that $x \in J(A)$.

Clearly, if A is archimedean, then $J(A) = 0$. Suppose now that $J(A) = 0$. Let $0 \leq a, b \in A$ and suppose that $na \leq b$ for all $n \in \mathbb{N}$. Then $na \leq b \leq b \vee 1 = c$ for all $n \in \mathbb{N}$. Since $c \geq 1$ and A has bounded inversion, $c^{-1} \in A$. Then $(na)c^{-1} =$

$n(ac^{-1}) \leq 1$ for all $n \in \mathbb{N}$. Therefore, $ac^{-1} \in J(A) = 0$, so that $a = 0$, and A is archimedean.

QED

It should be pointed out for later use that the containment $J(A) \subset \{x \in A : |x| \ll 1\}$ obtains without the assumption that A is bounded.

We will need the following lemma which characterizes the clopen subsets of $Max(A)$.

Lemma 2.3.6 *Let A be a commutative semi-prime f-ring with identity and bounded inversion. Then $\mathcal{K} \subset Max(A)$ is clopen if and only if $\mathcal{K} = \mathcal{M}_x$ where $x \in A$ is idempotent.*

PROOF

Suppose first that $\mathcal{K} = \mathcal{M}_x$ where $x \in A$ is idempotent. Then $\mathcal{M}_{x-1}$ is a basic open set disjoint to $\mathcal{M}_x$ with $\mathcal{M}_{x-1} \cup \mathcal{M}_x = Max(A)$. Therefore $\mathcal{M}_x$ is clopen. Suppose now that $\mathcal{K} \subset Max(A)$ is clopen. Since $\mathcal{K}$ is open, $\mathcal{K} = \bigcup_{x \in B} \mathcal{M}_x$ for some $B \subset A$. Since $\mathcal{K}$ is a closed subset of $Max(A)$, it is compact and therefore there exists $\{x_i : 1 \leq i \leq n\}$ such that $\mathcal{K} = \bigcup_{i=1}^n \mathcal{M}_{x_i}$. Since each $M \in Max(A)$ is convex, we may assume that each $x_i > 0$. Then $\mathcal{K} = \bigcup_{i=1}^n \mathcal{M}_{x_i} = \mathcal{M}_x$ where $x = \bigvee_{i=1}^n x_i$. By similar argument, since $\mathcal{K}$ is clopen, $Max(A) \setminus \mathcal{K} = \mathcal{M}_y$ for some $0 < y \in A$. Now $\mathcal{M}_x \cap \mathcal{M}_y = \mathcal{M}_{x \wedge y} = \emptyset$ so that $x \wedge y \in M$ for every $M \in Max(A)$. Let $x' = x - (x \wedge y)$ and $y' = y - (x \wedge y)$. Then $\mathcal{M}_x = \mathcal{M}_{x'}$, $\mathcal{M}_y = \mathcal{M}_{y'}$ and $x' \wedge y' = 0$. Since $\mathcal{M}_{x'} \cup \mathcal{M}_{y'} = \mathcal{M}_{x' \vee y'} = Max(A)$, $x' \vee y'$ is a multiplicative unit. Now let $e = x'(x' \vee y')^{-1}$ and $f = y'(x' \vee y')^{-1}$. Then $e \vee f = 1$, $e \wedge f = 0$ and $\mathcal{M}_e = \mathcal{M}_x$. Since $e \wedge f = 0$, $ef = 0$ so that $e^2 = e^2 \vee 0 = e^2 \vee ef = e(e \vee f) = e(1) = e$. Therefore, $\mathcal{K} = \mathcal{M}_e$ where e is idempotent.

QED

We now proceed to the proof of the main theorem.

PROOF

The proof will consist of two parts. We first will show that (1) implies (2); that (2) implies (3); that (3) is equivalent to (4), and that (3) implies (2). We will then show that if A is bounded that (3) implies (1).

Clearly, (1) implies (2). Suppose now that (2) holds. Since $Max(A)$ is compact, by Theorem 1.4.2, it suffices to show that $Max(A)$ is totally disconnected. Let $N, M \in Max(A)$ with $N \neq M$. Since A has bounded inversion, both M and N are ℓ-ideals. Pick $0 < x \in M \setminus N$. By the maximality of N, the ideal generated by x and N, $\langle N, x\rangle = A$. In particular, there exists $0 \neq y \in N$, $a \in A$ such that $y + ax = 1$. Then $y \notin M$, and $(y \vee 0) + ax = (y + ax) \vee (0 + ax) \geq y + ax = 1$ so that by bounded inversion, $(y \vee 0) + ax = u$ is a multiplicative unit in A. By the convexity of N and M, $y \vee 0 \in N \setminus M$. By a suitable relabelling, we have produced $0 < x \in M \setminus N$, $0 < y \in N \setminus M$ with $(x, y) = 1$. Now consider $x - yt^2 \in A[t]$. This satisfies the hypothesis of (2) and so there is an $a \in A$ so that $x - ya^2 = u$ is a multiplicative unit in A. Let $u^+ = u \vee 0$ and $u^- = (-u) \vee 0$. Then $u = u^+ - u^-$ and $u^+ \wedge u^- = 0$. Recall that the basic open sets of $Max(A)$ are of the form $\mathcal{M}_x = \{M \in Max(A) : x \notin M\}$ for $x \in A$. We now note that since the maximal ideals are ℓ-ideals, if $0 \leq x, y \in A$, then $\mathcal{M}_x \cap \mathcal{M}_y = \mathcal{M}_{x \wedge y}$ and $\mathcal{M}_x \cup \mathcal{M}_y = \mathcal{M}_{x \vee y}$. Now, $u^+ \vee u^- \geq u$ and as A has bounded inversion $u^+ \vee u^-$ is a multiplicative unit. Then $\mathcal{M}_{u^+} \cup \mathcal{M}_{u^-} = \mathcal{M}_{u^+ \vee u^-} = Max(A)$ and $\mathcal{M}_{u^+} \cap \mathcal{M}_{u^-} = \mathcal{M}_{u^+ \wedge u^-} = \emptyset$. Therefore $\{\mathcal{M}_{u^+}, \mathcal{M}_{u^-}\}$ is a clopen partition of $Max(A)$. Now, $-ya^2 \leq x - ya^2 \leq x$ so that $0 \leq u^+ \leq x$ and $0 \leq u^- \leq ya^2$. Then as $x \in M$ and M is convex, $u^+ \in M$ so that $u^- \notin M$ and therefore $M \in \mathcal{M}_{u^-}$. Similarly, since $y \in N$, it follows that $N \in \mathcal{M}_{u^+}$. Thus, for any two distinct points in $Max(A)$ there is a partition of $Max(A)$ into two clopen subsets each containing exactly one of the points. Therefore, $Max(A)$ is totally disconnected.

Suppose now that (3) holds. That is, $Max(A)$ is zero-dimensional. Let $\phi : Max(A) \to Max(S(A))$ be the map defined in Theorem 2.1.1. Then ϕ is continuous and onto, and since $S(A)$ is hyperarchimedean, $\phi(M) = M \cap S(A)$ for all $M \in Max(A)$. Since both of $Max(A)$ and $Max(S(A))$ are compact Hausdorff, it suffices to show that ϕ is one-to-one. Let $M \neq N \in Max(A)$. Since $Max(A)$ is zero-dimensional, hence totally disconnected, there is a clopen set $\mathcal{K} \subset Max(A)$ with $M \in \mathcal{K}$, $N \notin \mathcal{K}$. Since $\mathcal{K}$ is clopen, by Lemma 2.3.6, there is an $e \in A$, idempotent such that $\mathcal{K} = \mathcal{M}_e$. Since e is idempotent, $e \in S(A)$ so that $e \notin M \cap S(A)$ and $e \in N \cap S(A)$. Therefore $\phi(M) \neq \phi(N)$, and therefore ϕ is one-to-one.

If $Max(A) \cong Max(S(A))$ then, since $S(A)$ is hyper-archimedean, $Max(S(A)) = Min(S(A))$, and $Min(S(A))$ is zero-dimensional, so that $Max(A)$ is as well.

That (3) implies (2) is a special case of the second part of the proof. Suppose it has already been shown that if $A = A(1)$ then all four conditions are equivalent. Suppose now that (3) holds, that is $Max(A)$ is zero-dimensional and A is not necessarily bounded. Let $a - bt^2 \in A[t]$ be primitive with positive coefficients. Then there are $c, d \in A$ such that $ac + bd = 1$. Then $|1| = |ac + bd| \leq |ac| + |bd| = a|c| + b|d| \leq (a+b)|c| + (a+b)|d| = (a+b)(|c|+|d|)$. Since A has bounded inversion, $(a+b)(|c|+|d|)$ is a unit, so that $(a + b)$ is as well. Let $a' = a(a + b)^{-1}$ and $b' = b(a + b)^{-1}$. Then $a' - b't^2 \in A(1)[t]$ and $a' + b' = 1$ so that $a' - b't^2$ is primitive with positive coefficients in $A(1)$. If $Max(A)$ is zero-dimensional, by Corollary 2.2.1, $Max(A(1))$ is also. Therefore there is an $x \in A(1)$ such that $a' - b'x^2 = u$ is a multiplicative unit in $A(1)$. Since $A(1) \subset A$, $x \in A$ and u is a multiplicative unit in A. Therefore $a - bx^2 = u(a+b)$ is a multiplicative unit in A.

Now let us prove that (4) implies (1) for bounded rings. By applications of Lemmas 2.3.1, 2.3.2, 2.3.4, we may without loss of generality assume that A is archimedean. By Lemma 2.3.4, $Yos(A, 1) = Max(A)$ and by the preliminaries

on the Yosida embedding of an archimedean f-ring, A embedds as an f-subring $\hat{A} \subset D(Max(A))$, where $\hat{A}$ is an f-ring such that $1 \mapsto 1$. Since A is bounded, the Yosida embedding is actually into $C(Max(A))$. Let $Max(A) = X$. Identifying A with its image in $C(X)$, we have that A is an f-subring of $C(X)$ with $1 \in A$. Let $a_0 + a_1 t + \cdots + a_n t^n \in A[t]$ be primitive. If we consider the a_i as elements of $C(X)$, we have that $\bigcap_{i=1}^{n} Z(a_i) = \emptyset$. Then $\{coz(a_i) : 1 \leq i \leq n\}$ is a basic cover of X (in the sense of Chapter 16 [15]). Since X is compact and zero-dimensional, there exists a refinement of this basic cover by a partition of X. That is, there exists $\{\mathcal{K}_i : 1 \leq i \leq n\}$, a pairwise disjoint collection of clopen subsets of X, with $\mathcal{K}_i \subset coz(a_i)$ and such that $\bigcup_{i=1}^{n} \mathcal{K}_i = X$.

Define $f : \mathcal{K}_i \times \mathbb{R} \to \mathbb{R}$ by $f(x,r) = a_0(x) + a_1(x)r + \cdots + a_n(x)r^n$. Fix $x_0 \in \mathcal{K}_i$. Then as $\mathcal{K}_i \subset coz(a_i)$, $a_i(x) \neq 0$, so that $x \in {a_i}^{-1}(\mathbb{R} \setminus \{0\})$. Let $f_{x_0} : \mathbb{R} \to \mathbb{R}$ be defined by $f_{x_0}(r) = f(x_0, r)$. Then f_{x_0} is continuous, so that ${f_{x_0}}^{-1}(\mathbb{R} \setminus \{0\})$ is a non-empty open subset of $\mathbb{R}$. Therefore there is an $0 \neq r_0 \in {f_{x_0}}^{-1}(\mathbb{R} \setminus \{0\}) \cap \mathbb{Q}$. Now define $f_{r_0} : \mathcal{K}_i \to \mathbb{R}$ by $f_{r_0}(x) = f(x, r_0)$. Then f_{r_0} is continuous, and by construction, $f_{r_0}(x_0) \neq 0$. Therefore, $coz(f_{r_0}) \neq \emptyset$. Recall that r_0 is determined by x_0, so for ease of notation, let $coz(f_{r_0}) = N_{x_0}$. Then $\{N_x : x \in \mathcal{K}_i\}$ is an open cover of $\mathcal{K}_i$. Since $\mathcal{K}_i$ is compact, there exists $\{N_{x_j} : 1 \leq j \leq m\}$ such that $\mathcal{K}_i = \bigcup_{j=1}^{m} N_{x_j}$. Since $\mathcal{K}_i$ is a clopen subset of a compact zero-dimensional space X, $\mathcal{K}_i$ is a compact zero-dimensional space. Therefore there is a refinement of the basic cover $\{N_{x_j} : 1 \leq j \leq m\}$ by a partition of $\mathcal{K}_i$. Let $\{\mathcal{E}_j : 1 \leq j \leq m\}$ be this partition.

Recall that $X = Max(A)$. Since each $\mathcal{E}_j$ is clopen in X by Lemma 2.3.6, there is in $e_j \in A$ idempotent such that $\mathcal{E}_j = \mathcal{M}_{e_j}$. In the identification of A with $\hat{A}$, $e_j = \chi_{\mathcal{M}_{e_j}}$ Define $s_i : X \to \mathbb{R}$ by $s_i(x) = r_1 e_1(x) + \cdots + r_m e_m(x)$. Then s_i is continuous. Repeat this construction for each $\mathcal{K}_i$, and put $s = s_1 + \cdots + s_n$. Then $s \in C(X)$ and s is a

linear combination of rational multiples of idempotents of A. Since A has bounded inversion, $s \in A$.

Now consider $a_0 + a_1 s + \cdots + a_n s^n$ as an element of $C(X)$. By the construction of s, $Z(a_0 + a_1 s + \cdots + a_n s^n) = \emptyset$. If as an element of A, $a_0 + a_1 s + \cdots + a_n s^n \in M$ for any $M \in Max(A)$, then the Yosida embedding of A in $C(X)$ would give us an $x \in X$, namely M, with $a_0(x) + a_1(x)s(x) + \cdots + a_n(x)s^n(x) = 0$. Therefore $a_0 + a_1 s + \cdots + a_n s^n \notin M$ for every $M \in Max(A)$, so that $a_0 + a_1 s + \cdots + a_n s^n$ is a unit in A. Therefore A is local-global.

In the case that A is a vector lattice, nothing is changed by taking $S_{\mathbb{R}}(A)$ in the place of $S(A)$.

QED

We should note here that one interesting aspect of Theorem 2.3.1 is that condition (2) is a first order condition in the language of commutative semi-prime rings with identity; that is, condition (2) can be written in terms of universal and existential quantifiers and a finite number of elements, operations and relations.

We should also note that in the above proof for A archimedean, bounded, the substitution $s \in A$ such that $a_0 + a_1 s + \cdots + a_n s^n$ is a unit was an element of $S(A)$. This still holds if we drop the assumption of archimedeaneity.

<u>Corollary 2.3.1</u> *If A is a bounded commutative semi-prime f-ring with identity and bounded inversion which is local-global and $a_0 + a_1 t + \cdots + a_n t^n \in A[t]$ is primitive, then there is an $s \in S(A)$ such that $a_0 + a_1 s + \cdots + a_n s^n$ is a unit in A.*

PROOF

It suffices to show that an idempotent in $A/J(A)$ lifts to an idempotent in A. That is, if $x^2 + J(A) = x + J(A)$, then there is an $e \in A$ idempotent such that $x + J(A) = e + J(A)$. So, suppose that $x \in A$ is such that $x^2 + J(A) = x + J(A)$.

Then $x^2 - x = x(x-1) \in J(A)$, so that $x(x-1) \in M$ for every $M \in Max(A)$. Since M is a proper prime ideal this gives us that $x \in M$ or $x - 1 \in M$, but not both. Therefore, $\mathcal{M}_x$ and $\mathcal{M}_{x-1}$ partition $Max(A)$, and $\mathcal{M}_x$ is clopen. By Lemma 2.3.6, $\mathcal{M}_x = \mathcal{M}_e$ where $e \in A$ is idempotent. Suppose that $x + J(A) \neq e + J(A)$. Then there exists an $M \in A$ such that $x - e \notin M$. Since $\mathcal{M}_x = \mathcal{M}_e$ we have that $x \notin M$ and $e \notin M$. But $x(x-1), e(e-1) \in M$, so $x - 1, e - 1 \in M$. A contradiction as $x - e = (x-1) - (e-1) \notin M$. Therefore $x + J(A) = e + J(A)$.

QED

<u>Corollary 2.3.2</u> *Let A be a bounded commutative semiprime f-ring with identity and bounded inversion and let B be an f-subring of A with bounded inversion and $1 \in B$. If A is local global and $S(A) \subset B$, then B is local-global.*

PROOF

Since A is bounded and B is an f-subring of A, B is bounded. By Theorem 2.3.1, $Max(A) \cong Max(S(A))$. Let ϕ be this homeomorphism. By Theorem 2.1.1, and Lemma 2.2.1, the maps $\psi : Max(A) \to Max(B)$ and $\theta : Max(B) \to Max(S(A))$ are continous surjections with $\theta \circ \psi = \phi$. Therefore ψ is one-to-one and hence a homeomorphism. Then $Max(B)$ is zero-dimensional and so, by Theorem 2.3.1, B is local-global.

QED

<u>Corollary 2.3.3</u> *If A is a commutative semi-prime f-ring with identity and bounded inversion which is in addition archimedean, bounded and local-global, then $C(Max(A))$ is the largest bounded archimedean local-global extension B of A such that A separates the points of B.*

PROOF

We first note that for A as above, $C(Max(A))$ is a commutative semi-prime f-ring with identity and bounded inversion which is in addition archimedean, bounded and local-global. Since $Max(A)$ is compact Hausdorff, $Max(C(Max(A))) \cong \beta(Max(A)) \cong Max(A)$, so that A separates the points of $C(Max(A))$. Suppose now that B is a bounded, archimedean, local-global extension of A with $Max(A) \cong Max(B)$. Then as in the proof of Theorem 2.3.1, B is an f-subring of $C(Max(B))$. Since $Max(A) \cong Max(B)$, the result follows.

QED

Now let $A = C(X)$ be the ring of continuous real valued functions on X. We will denote $S_{\mathbb{R}}(C(X))$ by $S(X)$. We make the following observations.

1. $C(X)$ is a commutative semi-prime f-ring with identity and bounded inversion.

2. If X is compact, then $C(X)$ is bounded.

3. If X is compact Hausdorff, then $X \cong \beta X \cong Max(C(X))$.

These, together with Theorem 2.3.1, give us the following corollary.

<u>Corollary 2.3.4</u> *If X is a compact Hausdorff space, the following are equivalent:*

1. *$C(X)$ is local-global.*

2. *If $f - gt^2 \in C(X)[t]$ is primitive with positive coefficients, then $f - gt^2$ represents a unit in $C(X)$.*

3. *X is zero-dimensional.*

4. *$X \cong Max(S(X))$.*

One of the troublesome aspects of the above results is that we get equivalence of the stated conditions for bounded rings or in the case of $C(X)$, for X compact. We now consider a class of rings for which the assumption of boundedness can be dropped. We need the following definitions.

Definition 2.3.2 *A ring A with identity is said to be a Bezout ring if every finitely generated ideal is principal.*

Definition 2.3.3 *A Tychonoff space X is said to be an F-space if every dense cozero is C^* embedded.*

Bezout rings and Bezout domains are discussed extensively throughout R. Gilmer's "Multiplicative Ideal Theory" [16]. The following is due to Gillman and Henriksen [14] and in the present form is Theorem 14.25 [15].

Theorem 2.3.2 *Let X be a Tychonoff space. Then the following are equivalent.*

1. *X is an F-space.*

2. *βX is an F-space.*

3. *$C(X)$ is a Bezout ring.*

4. *$0 \leq a \leq b \in C(X)$ implies that $a = bf$ for some $f \in C(X)$.*

5. *Every ideal in $C(X)$ is convex.*

6. *Every ideal in $C(X)$ is an ℓ-ideal.*

7. *For $0 \leq a, b \in C(X)$, the ideal $(a, b) = (a + b)$.*

8. *For $f \in C(X)$, there exists a $k \in C(X)$ such that $f = k|f|$.*

9. *The localization of $C(X)$ at any maximal ideal is a valuation ring.*

We recall here that an integral domain D is called a *valuation ring* if the ideals of D form a chain. Recall also that for A a commutative semi-prime f-ring with identity and bounded inversion we have identified the localization of A at a maximal ideal M with the quotient ring $A/O(M)$. It is in this sense that (9) above should be interpreted. The following is a partial generalization of the above to semi-prime f-rings with bounded inversion [25].

<u>Theorem 2.3.3</u> *Let A be a semi-prime f-ring with bounded inversion. Then the following are equivalent.*

1. *A is a Bezout ring.*

2. *$0 \leq a \leq b \in A$ implies that $a = bf$ for some $f \in A$.*

3. *Every ideal of A is convex.*

4. *Every ideal of A is an ℓ-ideal.*

5. *For $0 \leq a, b \in A$, the ideal $(a, b) = (a + b)$.*

6. *The localization of A at any maximal ideal is a valuation ring.*

We should note that if A is Bezout, then by (5) above, $(f^+, f^-) = (f^+ + f^-) = (|f|)$. So that in particular, $f = k|f|$ for some $f \in A$. For Bezout rings, Theorem 2.3.1 improves to the following.

<u>Theorem 2.3.4</u> *Let A be a commutative semi-prime f-ring with identity and bounded inversion. If in addition, A is a Bezout ring then the following are equivalent.*

1. A is a local-global ring.

2. If $a - bt \in A[t]$ is primitive, then $a - bt$ represents a unit.

3. $Max(A)$ is zero-dimensional.

PROOF

Clearly (1) implies (2). Suppose now that (2) holds. Since $Max(A)$ is compact, it suffices to show that $Max(A)$ is totally disconnected. Suppose that $M \neq N \in Max(A)$. As in the proof of Theorem 2.3.1, there exists $0 < a \in M \setminus N$ and $0 < b \in N \setminus M$ with $a \wedge b = 0$. Let $f = a - b$. Then $f^+ = a$ and $f^- = b$. By Theorem 2.3.3, since A is Bezout, there is a $k \in A$ such that $f = k|f|$. Then $a - b = k(a + b) = ka + kb$, so that $(1 - k)a = (1 + k)b$. Since $a \wedge b = 0$, we have that $(1 - k)a = (1 + k)b = 0$, and so $a = ka$ and $-b = kb$. Let $g = 1 - k^2$. Then $ga = gb = 0$, so that $g \in N$ and $g \in M$. Also, $g + k^2 = 1$, so by the hypothesis there is a $v \in A$ such that $k - gv = u$ is a multiplicative unit. Consider now the basic open sets $\mathcal{M}_{u^+}$ and $\mathcal{M}_{u^-}$. Since u is a multiplicative unit, these sets partition $Max(A)$. Now, $u^+a = (u \vee 0)a = ua \vee 0 = (k - gv)a \vee 0 = ka \vee 0 = a \vee 0 = a$, so that $u^+ \notin N$ lest $a \in N$. Similarly, $u^-b = b$, so that $u^- \notin M$. Therefore $N \in \mathcal{M}_{u^+}$ and $M \in \mathcal{M}_{u^-}$, and as these are disjoint, $Max(A)$ is totally disconnected.

Suppose now that (3) holds. Let $a_0 + a_1t + \cdots + a_nt^n \in A[t]$ be primitive. Then there exists $b_0, b_1, \ldots, b_n \in A$ such that $a_0b_0 + a_1b_1 + \cdots gg + a_nb_n = 1$. Then $1 = |a_0b_0 + a_1b_1 + \cdots + a_nb_n| \leq (|a_0| + |a_1| + \cdots + |a_n|)(|b_0| + |b_1| + \cdots + |b_n|)$. Since A has bounded inversion, $|a_0| + |a_1| + \cdots + |a_n| = a$ is a multiplicative unit. Then, $a_0a^{-1} + a_1a^{-1}t + \cdots + a_na^{-1}t^n \in A(1)[t]$. Since A is Bezout, for each a_i, $0 \leq i \leq n$, there is a k_i such that $a_i = k_i|a_i|$. But then $k_ia_i = |a_i|$ and we may assume without loss of generality that $|k_i| \leq 1$ (take $k' = (k \wedge 1) \vee -1$). Then $k_0a_0a^{-1} + k_1a_1a^{-1} + \cdots + k_na_na^{-1} = 1$ and therefore $a_0a^{-1} + a_1a^{-1}t + \cdots + a_na^{-1}t^n \in A(1)[t]$ is primitive. Since

$Max(A)$ is zero-dimensional, by Corollary 2.2.1, $Max(A(1))$ is as well. Applying Theorem 2.3.1 to $Max(A(1))$ we have that $A(1)$ is local-global. Therefore, there exists a $v \in A(1)$ such that $a_0a^{-1} + a_1a^{-1}v + \cdots + a_na^{-1}v^n = u$ is a multiplicative unit in $A(1) \subset A$. Therefore $a_0 + a_1v + \cdots + a_nv^n = au$ is a multiplicative unit in A. Since $v \in A(1) \subset A$, A is local-global.

QED

For X a Tychonoff space we have the following corollary.

<u>*Theorem 2.3.5*</u> *If X is an F-space, then the following are equivalent*

1. *$C(X)$ is local-global.*

2. *If $f - gt \in C(X)[t]$ is primitive, then it represents a unit.*

3. *X is strongly zero-dimensional.*

PROOF

By Theorem 2.3.2, X is an F-space if and only if $C(X)$ is a Bezout ring. That (1) implies (2) and that (2) implies (3) then follows directly from Theorem 2.3.4 and the observation that $Max(C(X)) \cong \beta X$. To prove that (3) implies (1), suppose that X is strongly zero-dimensional; that is βX is zero-dimensional. Then $C(\beta X)$ is local-global. But, $C(\beta X) \cong C^*(X)$ the ring of bounded continuous real valued functions on X. The proof now follows as in the proof of Theorem 2.3.4 taking $C(X) = A$ and $C^*(X) = A(1)$.

QED

Recall in the statement of Theorem 2.3.1, that conditions (2), (3) and (4) are always equivalent and that (1) implies (2). The assumption of boundedness was needed only in the proof that (3) implies (1). The problem that prevented this implication in the general case is one of "cutting down" a primitive polynomial in

$f(t) \in A[t]$ to a polynomial $\tilde{f}(t) \in A(1)[t]$ and maintaining primitivity. The most general result in this context is the following theorem.

Theorem 2.3.6 If A is a commutatuve semi-prime f-ring with identity and bounded inversion, then the following are equivalent.

1. *Every primitive polynomial $f(t) \in A[t]$ having coefficients that are comparable to zero represents a multiplicative unit.*

2. *$Max(A)$ is zero-dimensionl.*

PROOF

It suffices to show that a primitive polynomial having coefficients that are comparable to zero in $A[t]$ differs from a primitive polynomial in $A(1)[t]$ by a multiplicative unit of A. We can then apply Theorem 2.3.1. The proof now proceeds exactly as in the proof of Theorem 2.3.4 taking the k_i to be 1 if $a_i \geq 0$ or -1 if $a_i \leq 0$.

QED

CHAPTER 3
QUASI-SPECKER f-RINGS AND f-RINGS OF SPECKER TYPE

3.1 Introduction

In the previous chapter, we have seen that if $S(A)$ (respectively $S_{\mathbb{R}}(A)$) is a convex f-subring of the commutative semi-prime f-ring A, then (i) $S(A) = A$ (respectively $S_{\mathbb{R}}(A) = A$), and (ii) A is local-global. In this chapter we will explore two other distinct types of containments of $S(A)$ in A and the effects of these containments on the structure of A. One of these containments is based on the ring structure and the other on the order structure of A. We need the following definitions.

Definition 3.1.1 *Let R be a commutative ring. A subring S of R is said to be large if for every non-zero S-submodule I of R, $I \cap S \neq 0$. R is also called an S-essential extension of S.*

Definition 3.1.2 *Let H be an ℓ-group, G an ℓ-subgroup of H. G is said to be large in H if for every non-zero convex ℓ-subgroup C of H, $C \cap G \neq 0$. H is also called an essential extension of G.*

Since the above definitions become ambiguous in the context of f-rings, we will adopt the convention that whenever using definitions dependent on the lattice structure we will precede them by an "o". Thus if A is an f-ring and B is an f-subring such that B is large in A as an ℓ-subgroup, we will say that B is *o-large* in A, or that A is an *o-essential* extension of B.

We will make extensive use of the following well known characterization of essential extensions.

<u>*Theorem 3.1.1*</u> *Let R be a commutative ring, S a subring of R. Then, S is large in R if and only if for every $0 \neq a \in R$, there is a $b \in S$ such that $0 \neq ab \in S$.*

PROOF

Suppose that S is large in R. Let $0 \neq a \in R$ and let $< a >= \{sa : s \in S\}$ be the S-submodule of R generated by a. Then $< a > \cap S \neq 0$, so there is a $b \in S$ such that $0 \neq ab \in S$. Conversly, if $0 \neq I$ is an S-submodule of R, then there is an $0 \neq a \in I \subset R$. By the hypothesis, there is a $b \in S$ such that $0 \neq ab \in S$. Since I is an S-submodule, $ab \in I$.

QED

Recall now from Chapter One that for an ℓ-group G, $\mathcal{C}(G)$ is the collection of ℓ-ideals of G, and that $\mathcal{C}(G)$ is a complete, distributive, brouwerian sublattice of the lattice of all subgroups of G.

The next characterization of the o-large f-subrings of an f-ring will depend on a certain meet subsemi-lattice of $\mathcal{C}(G)$. We first need to set up some machinery.

<u>*Definition 3.1.3*</u> *Let G be an ℓ-group. For any subset $T \subset G$, let $T^{\perp} = \{a \in G : |a| \wedge |t| = 0 \ \forall t \in T\}$. $T^{\perp}$ is called the polar of T in G.*

We will use the following notational conventions. For $g \in G$, $g^{\perp} = \{g\}^{\perp}$ and for any subset $T \subset G$, $T^{\perp\perp} = \{T^{\perp}\}^{\perp}$. $g^{\perp\perp}$ is called the *principal polar* of g.

<u>*Definition 3.1.4*</u> *Let G be an ℓ-group. $P \in \mathcal{C}(G)$ is said to be a* polar *of G if $P = T^{\perp}$ for some subset $T \subset G$.*

If we consider $\perp$ as a unary operation on $\mathcal{C}(G)$ then the following results are well known and can be found in particular in Chapter 1 of "Lattice-Ordered Groups" [2].

1. $B \subset B^{\perp\perp}$.

2. If $B \subset C$, then $C^{\perp} \subset B^{\perp}$.

3. $B^{\perp} = B^{\perp\perp\perp}$.

4. $B^{\perp} \cap C^{\perp} = (B \vee C)^{\perp}$ where $\vee$ is the supremum in the lattice $\mathcal{C}(G)$.

5. $P \in \mathcal{C}(G)$ is a polar if and only if $P = P^{\perp\perp}$.

6. For any subset $T \subset G$, $T^{\perp} \in \mathcal{C}(G)$.

Let $\mathcal{P}(G) = \{P \in \mathcal{C}(G) : P^{\perp\perp} = P\}$. The following is Theorem 1.2.5 [2]. That $\mathcal{P}(G)$ is a complete boolean algebra is originally due to F. Šik [31] (1960). The mapping context is part of a more general lattice theoretic result due to V. Glivenko [18] (1929).

Theorem 3.1.2 *$\mathcal{P}(G)$ is a complete boolean algebra when equipped with the meet $\cap$, the join $\sqcup$ defined by $B \sqcup C = (B \vee C)^{\perp\perp}$ and complementation $\perp$. Furthermore the map $B \mapsto B^{\perp\perp}$ is a lattice epimorphism from $\mathcal{C}(G)$ to $\mathcal{P}(G)$.*

With this, the characterization we need is the following generalization of Theorem 11.1.15 [5]. These results are originally due to P. Conrad [7].

Theorem 3.1.3 *Let G be an ℓ-subgroup of the ℓ-group H. Then each of the following implies the next.*

1. *G is o-large in H.*

2. *For every $0 < h \in H$, there is a $0 < g \in G$ and an $n \in \mathbb{N}$ such that $g \leq nh$.*

3. *For each nonzero $P \in \mathcal{P}(H)$, $P \cap G \neq 0$.*

4. For each $0 \neq h \in H$ *there exists a* $0 \neq g \in h^{\perp\perp} \cap G$.

5. The map $P \mapsto P \cap G$ *is a boolean isomorphism from* $\mathcal{P}(H)$ *to* $\mathcal{P}(G)$.

Furthermore, 1 and 2 are always equivalent; 3,4 and 5 are always equivalent, and if H *is archimedean then 3 implies 1.*

Now that we have some useful characterizations of essential and o-essential extensions we need to look at some particular extensions in both contexts. We will first deal with the o-essential extensions. The following depends only on the ℓ-group structure.

For an ℓ-group G we will be interested in several o-extensions of G; the lateral completion, the orthocompletion and, in the case where G is archimedean, the Dedekind completion and the o-essential closure. We need the following definitions and notation.

Definition 3.1.5 *Let* G *be an* ℓ*-subgroup of the* ℓ*-group* H. G *is* dense *in* H *if for every* $0 < h \in H$, *there is a* $0 < g \in G$ *such that* $g \leq h$.

Clearly G is dense in H implies that G is o-large in H.

Definition 3.1.6 *An* ℓ*-group is* Dedekind complete *if every collection of elements which is bounded above has a supremum. For* G *archimedean, an* ℓ*-group* $G^{\wedge}$ *is a* Dedekind completion *of* G *if* G *is an* ℓ*-subgroup of* $G^{\wedge}$, $G^{\wedge}$ *is Dedekind complete and each element of* $G^{\wedge}$ *is the supremum of elements of* G.

Definition 3.1.7 *An* ℓ*-group is* laterally complete *if every collection of pairwise disjoint elements has a supremum. An* ℓ*-group* G^{L} *is the* lateral completion *of* G *if* G *is a dense* ℓ*-subgroup of* G^{L}, G^{L} *is laterally complete and no proper* ℓ*-subgroup of* G^{L} *contains* G *and is laterally complete.*

<u>*Definition 3.1.8*</u> *Let G be an ℓ-group. $H \in \mathcal{C}(G)$ is said to be a cardinal summand of G if there is a $K \in \mathcal{C}(G)$ with $H \wedge K = 0$ and $H \vee K = G$. We denote this by $G = H \boxplus K$.*

Cardinal summands, in addition to being direct summands are unique. Also, if $G = H \boxplus K$ then both H and K are ℓ-groups, and if $g = h + k$ with $h \in H$, $k \in K$, then $0 \leq g$ if and only if $0 \leq h$ and $0 \leq k$.

<u>*Definition 3.1.9*</u> *An ℓ-group G is called strongly projectable if for each $P \in \mathcal{P}(G)$, $G = P \boxplus P^{\perp}$. G is called projectable if for each $g \in G$, $G = g^{\perp} \boxplus g^{\perp\perp}$.*

<u>*Definition 3.1.10*</u> *An ℓ-group is* orthocomplete *if it is laterally complete and projectable. An ℓ-group G^{o} is an* orthocompletion *of G if G is dense in G^{o}, G^{o} is orthocomplete and no proper ℓ-subgroup of G^{o} contains G and is orthocomplete.*

<u>*Definition 3.1.11*</u> *An ℓ-group is* o-essentially closed *if it admits no proper o-essential extensions. An* o-essential closure *of an ℓ-group is an o-essentially closed o-essential extension.*

It should be pointed out that unless we restrict our attention to archimedean ℓ-groups an ℓ-group always admits a proper o-essential extension because any ℓ-group can be lexicographically extended and will be large in this extension. For this reason, the idea of an o-essential closure only makes sense in the category of archimedean ℓ-groups. The lateral completion of an ℓ-group is always unique, if in addition G is an archimedean ℓ-group then $G^{\wedge}$ and G^{o} are unique as well.

Recall, that for an ℓ-group G, $\mathcal{P}(G)$ is a complete boolean algebra. Let $X = St(\mathcal{P}(G))$ be the Stone dual of $\mathcal{P}(G)$. Since $\mathcal{P}(G)$ is complete, X is compact, Hausdorff and extremally disconnected. Recall that

$$D(X) = \{f : X \to \mathbb{R} \cup \{\pm\infty\} : f \text{ is continuous and } f^{-1}(\mathbb{R}) \text{ is dense open}\}$$

For $f, g \in D(X)$, $Y = f^{-1}(\mathbb{R}) \cap g^{-1}(\mathbb{R})$ is a dense subset of X. Since X is extremally disconnected, Y is C^* embedded in X. Therefore $\beta Y = \beta X = X$. We have $f + g : Y \to \mathbb{R} \cup \{\pm\infty\}$ with $\mathbb{R} \cup \{\pm\infty\}$ compact. Therefore there exists a unique extinsion of $f+g$ to $\beta Y = X$. We then define the group element $f+g$ to be this extension. We then have that, for X extremally disconnected, $D(X)$ with the pointwise ordering is an ℓ-group. We can also define pointwise multiplication as above making $D(X)$ a ring. In fact, $D(X)$ is a complete vector lattice. The following theorem is due to S. Bernau [4]. The earlier work of Pinsker and Vulich [32] obtains a similar result for complete vector lattices.

<u>Theorem 3.1.4</u> *If G is an archimedean ℓ-group then there is an ℓ-isomorphism π of G onto a large ℓ-subgroup of $D(X)$ that preserves all joins and meets. If G is a vector lattice, π also preserves scalar multiplication. Furthermore, if $\{e_\lambda : \lambda \in \Lambda\}$ is a maximal disjoint subset of G, π can be chosen so that $\{\pi(e_\lambda) : \lambda \in \Lambda\}$ is a set of characteristic functions of a family of pairwise disjoint clopen subsets of X whose union is dense. In particular, if e is a weak order unit, then π can be chosen so that $\pi(e)$ is the characteristic function of X and if e is a strong order unit, then $\pi(G) \subset C(X)$. Finally if π_1 and π_2 are any two such embeddings, then there is a $d \in D(X)$ such that $d\pi_1(g) = \pi_2(g)$ for all $g \in G$.*

The following is Theorem 3.4 [7] and it provides us with an "umbrella" extension.

<u>Theorem 3.1.5</u> *Each archimedean ℓ-group G admits a unique o-essential closure G^e. G^e is the Dedekind complete, laterally complete, divisible ℓ-group in which G is o-large and G^e is ℓ-isomorphic to $D(X)$ where X is the Stone dual of the boolean algebra of polars of G.*

We now turn our attention to the ring theoretic aspects of essential extensions. Recall that the classical ring of quotients of A is $qA = S^{-1}A$, where S is the set of regular elements of A. We need the following concept which is due to Y. Utumi [30].

<u>Definition 3.1.12</u> *Suppose that A is a subring of the ring B. B is called a ring of quotients of A if for every $b_1, b_2 \in B$, with $b_2 \neq 0$, there is an $a \in A$ such that $ab_1 \in A$ and $ab_2 \neq 0$.*

With this definition, it is clear that qA is a ring of quotients of A, and that any ring of quotients of A is an essential extension of A. It is not necessarily the case that qA is a maximal essential extension of A; the maximal object is the so-called *complete ring of quotients*, $Q(A)$. What follows are two quite different constructions of $Q(A)$. The first is due to J. Lambek and can be found in Chapter 2 [22].

<u>Definition 3.1.13</u> *An ideal I in a commutative ring A is said to be dense if for every $a \in A$, $aI = 0$ implies $a = 0$.*

Let A be a commutative ring and let $\mathrm{Hom}(D, A)$ be the set of A homomorphisms of D into A. Let $F(A) = \{f \in \mathrm{Hom}(D, A) : D \text{ is a dense ideal of } A\}$. We call $f \in F(A)$ a *fraction*. We can define addition and multiplication on $F(A)$ as follows.

For $f_1 \in \mathrm{Hom}(D_1, A)$, $f_2 \in \mathrm{Hom}(D_2, A)$,

1. $f_1 + f_2 \in \mathrm{Hom}(D_1 \cap D_2, A)$ is given by $(f_1 + f_2)(d) = f_1(d) + f_2(d)$.

2. $f_1 f_2 \in \mathrm{Hom}((f_2{}^{-1}(D_1)) \cap D_1, A)$ is given by $f_1 f_2(d) = f_1(f_2(d))$.

$F(A)$ is closed under the above operations since for D_1, D_2 dense, both $D_1 \cap D_2$ and $f_2{}^{-1}(D_1) \cap D_1$ are dense. Now define an equivalence relation on $F(A)$ by $f_1 \equiv f_2$ if $f_1 = f_2$ on some dense ideal D. For $f \in F(A)$ let $[f]$ denote the equivalence class of f. Let $Q(A) = \{[f] : f \in F(A)\}$. For $a \in A$, let f_a be multiplication

by a. If $[f_a] = 0$ then $aD = 0$ for some dense ideal D, so that $a = 0$. Therefore $a \mapsto f_a$ is an monomorphism of A in $Q(A)$. This mapping is called the *canonical monomorphism.* With the above operations suitably altered for equivalence classes we have the following theorem which is Proposition 1, §2.3 [22].

Theorem 3.1.6 *If A is a commutative ring, then $Q(A)$ is also a commutative ring. It extends A and will be called its complete ring of quotients.*

The following useful results can also be found in Lambek [22].

Theorem 3.1.7 *[Prop.6 §2.3[22]] Let A be a subring of the commutative ring B. Then the following are equivalent.*

1. *B is a ring of quotients of A (in the sense of Utumi).*
2. *For all $0 \neq b \in B$, $b^{-1}A$ is a dense ideal of A and $b(b^{-1}A) \neq 0$.*
3. *There exists a monomorphism of B into $Q(A)$ that extends the canonical monomorphism $a \mapsto f_a$ of A into $Q(A)$.*

Recall that an R-module M is said to be *injective* if A and B are R-modules and $\phi : A \to B$ is a monomorphism, then for any homomorphism $\psi : A \to M$ there is a $\kappa : B \to M$ such that $\kappa \circ \phi = \psi$. R is called *self-injective* if it is injective as an R-module. The following are all found in Chapter 4 [22].

Lemma 3.1.1 $Q(Q(A)) = Q(A)$.

If we restrict ourselves to the case where A is semi-prime, we have the following theorems.

Lemma 3.1.2 *If A is semi-prime, then $Q(A)$ is self-injective.*

<u>*Lemma 3.1.3*</u> *If A is semi-prime, then $Q(A)$ is the injective hull of A.*

Equivalently, $Q(A)$ is the maximal essential extension of A. This gives us the following lemma.

<u>*Lemma 3.1.4*</u> *If A is an essential extension of B, then $Q(A)$ is an essential extension of $Q(B)$.*

We now look at Banaschewski's construction of $Q(A)$ [3]. This is a generalization of the construction carried out in "Rings of Quotients of Rings of Functions" [12] where it is shown that if X is a Hausdorff space, then

$$Q(X) = Q(C(X)) = \varinjlim\{C(U) : U \subset X \text{ is dense, open}\}$$

Let A be a commutative semi-prime ring with identity. Let $\Omega \subset Spec(A)$ be such that $\bigcap \Omega = 0$. Call such a family of prime ideals a *seperating family*. Topologize Ω with the hull-kernel topology. Let $\mathcal{D} = \{U \subset \Omega : U \text{ is dense, open}\}$. For $U \in \mathcal{D}$, let $A_U = \{f \in \prod_{P \in U} q(A/P) :$ for every $Q \in U$ there is a neighborhood V with $Q \in V \subset U$ and there are $a, b \in A$ with $f(P') = (a + P')/(b + P')$ for every $P' \in V\}$. For each $U \in \Omega$, dense, open, A_U is a subring of $\prod_{P \in \Omega} q(A/P)$ containing A. If $U \subset V$ are dense, open in Ω, then the map $\pi_{V,U} : A_V \longrightarrow A_U$ given by $\pi_{V,U}(f) = f|_U$ is a ring homomorphism. It can then be shown that the rings A_U together with the maps $\pi_{V,U}$ form a directed system indexed by $I = \{U \in \mathcal{D}\}$ where $V \leq U$ if $U \subset V$. Banaschewski's result is the following

<u>*Theorem 3.1.8*</u> *Let A be a commutative semi-prime ring, and Ω a seperating family of primes. Then,*

$$Q(A) = \varinjlim(A_U, \pi_{V,U})$$

In particular, since A is semi-prime, if we take $\Omega = Min(A)$, then the canonical map $A \mapsto A/P \mapsto q(A/P)$ induces an imbedding

$$A \longrightarrow \prod_{P \in Min(A)} A/P \longrightarrow \prod_{P \in Min(A)} q(A/P)$$

Moreover, in the case that A is an f-ring, since each minimal prime ideal is an ℓ-ideal, each A/P is an ordered ring and hence each $q(A/P)$ is an ordered field and the canonical maps are all ℓ-homomorphisms. If we take the pointwise ordering on $\prod_{P \in Min(A)} A/P$ and $\prod_{P \in Min(A)} q(A/P)$, then these too are f-rings and the induced maps are ℓ-homomorphisms. That a ring A has a unique maximal essential extension was first shown by Y. Utumi [30]. That there is a canonical ordering on $Q(A)$, extending the ordering on A, for which $Q(A)$ is an f-ring was shown by F. Anderson [1]. The following theorem bringing together these results with Banaschewski's construction of $Q(A)$ is due to J. Martinez [24]. Recall that a ring A is a *Von Neumann regular* ring if for every $a \in A$ there is an $x \in A$ such that $axa = a$.

<u>*Theorem 3.1.9*</u> *For each semiprime f-ring A, the maximal ring of quotients $Q(A)$ admits a lattice-ordering making it a semi-prime f-ring and containing A as an f-subring. $f \in Q(A)$ is positive iff for each dense open subset W of $Min(A)$, and each $P \in W$ there exists an open set U and positive elements a and b in A, so that $P \in U \subset W$ and for each $Q \in U$, $f(Q) = (a+Q)/(b+Q)$. This is the unique lattice-ordering on $Q(A)$ making it an f-ring containing A as an f-subring. Finally, $Q(A)$ is a Von Neumann regular ring.*

3.2 Quasi-Specker f-Rings and f-Rings of Specker-Type

Let A be a commutative semi-prime f-ring with identy and bounded inversion. Recall that $S(A)$ is the $\mathbb{Q}$ subalgebra of A generated by the idempotents of A, and

that if A is an f-algebra, $S_{\mathbb{R}}(A)$ is the real subalgebra generated by the idempotents of A.

<u>Definition 3.2.1</u> *We say that A is a specker-type ring if $S(A)$ is a large subring of A. If A is an f-algebra such that $S_{\mathbb{R}}(A)$ is a large subring, then A is a specker-type algebra. We say that A is a quasi-specker ring if $S(A)$ is o-large in A.*

When A is an f-algebra, the distinction between being a specker-type ring and a specker-type algebra is unimportant in terms of the method and content of the proofs that follow. The significant difference is that the objects which characterize the two specker properties are possibly different. We don't need to make this distinction for the definition of quasi-specker, since in the case that A is an f-algebra, $S(A)$ is order large in $S_{\mathbb{R}}(A)$. Since many of the results are derived for the special case $A = C(X)$ we will call a Tychonoff space X a *specker space* or a *quasi-specker space* if $C(X)$ is a specker-type algebra or a quasi-specker ring respectively.

We will need the following definition.

<u>Definition 3.2.2</u> *A commutative semi-prime ring A is locally inversion closed if for every $a \in A$, and every $P \in Min(A)$ with $a \notin P$, there is an open neighborhood U, with $P \in U \subset \mathcal{N}_a$, and there exists $b \in A$ such that $ab = 1 \bmod Q$ for every $Q \in U$.*

The following characterization is due to J. Martinez [24].

<u>Theorem 3.2.1</u> *A semi-prime f-ring A is locally inversion closed if and only if for each $0 < a \in A$, there exists $\{c_i : i \in I\}$ such that $0 < c_i \leq a$ and $ax_i = c_i$ for some $x_i \in Ac_i$ and $a^{\perp} = \bigcap {c_i}^{\perp}$.*

<u>Theorem 3.2.2</u> *Let A be a commutative semi-prime f-ring with 1 and bounded inversion. If A is local-global and $J(A) = 0$ then A is locally inversion closed.*

PROOF

Let $0 < a \in A$. Then $\mathcal{M}_a$ is a nonempty open subset of $Max(A)$. Since A is local-global, by Theorem 2.3.1, $Max(A)$ is zero-dimensional and so has a base of clopen sets. Let $M \in \mathcal{M}_a$. Then there is a clopen $K_M \subset Max(A)$ with $M \in K_M \subset \mathcal{M}_a$. Since K_M is clopen, by Lemma 2.3.6, there is an $e_M \in A$ idempotent with $K_M = \mathcal{M}_{e_M}$. We claim that $e_M \in a^{\perp\perp}$; otherwise, there is $0 \leq x \in A$ with $x \wedge a = 0$ and $x \wedge e_M \neq 0$. Since $J(A) = 0$, there is an $N \in Max(A)$ with $x \wedge e_M \notin N$. Since N is an l-ideal, we have that $x \notin N$ and $e_M \notin N$. Then $N \in \mathcal{M}_{e_M} \subset \mathcal{M}_a$, so that $a \notin N$. Since $x \wedge a = 0$ and N is a prime l-ideal, $x \in N$; a contradiction. Therefore $e_M \in a^{\perp\perp}$. In particular, $e_M \wedge a \neq 0$.

Since e_M is idempotent we can write A as $A = Ae_M \boxplus A(1 - e_M)$. Considering Ae_M we have that this is both an ideal and an ℓ-ideal of A. Moreover, any ideal of Ae_M is an ideal of A.

Now let $P \in Max(Ae_M)$. Since $e_M \notin P$, the ideal generated in A by P and $1 - e_M$ is proper and so is contained in some $N \in Max(A)$ with $e_M \notin N$. Then $N \subset \mathcal{M}_{e_M} \subset \mathcal{M}_a$ so that $a \notin N$ as well. Therefore $ae_M \notin N$ as N is a prime ideal, and $P \subset N$ so that $ae_M \notin P$. We have shown that ae_M misses every maximal ideal in $Max(Ae_M)$ so that ae_M is a multiplicative unit in Ae_M. Therefore there is an $x_M \in Ae_M$ such that $ae_Mx_M = e_M$, the multiplicative identity in Ae_M.

Now for each $M \in \mathcal{M}_a$, let e_M and x_M be as above and let $c_M = ae_M$. We have

1. $0 < c_M \leq a$. This holds since $0 < e_M \leq 1$ and $e_M \wedge a \neq 0$.

2. $a(ae_Mx_M) = ae_M = c_M$ and $ae_Mx_M = c_Mx_M \in Ac_M$.

It remains to show that $a^\perp = \bigcap {c_M}^\perp$. Clearly $a^\perp \subset \bigcap {c_M}^\perp$ as c_M is a multiple of a. Suppose $0 < b \notin a^\perp$. Then $b \wedge a \neq 0$. Since $J(A) = 0$, there is a maximal ideal M with $b \wedge a \notin M$. By convexity, $b \notin M$ and $a \notin M$. Therefore $M \in \mathcal{M}_a$. As

shown above, there is an idempotent e_M with $M \in \mathcal{M}_{e_M} \subset \mathcal{M}_a$. Let $c_M = ae_M$. If $b \wedge c_M = 0$, since $b \notin M$ and M is prime, $c_M \in M$. Then $c_M e_M = a(e_M{}^2) = ae_M \in M$ so that $a \in M$ or $e_M \in M$. This is a contradiction since $M \in \mathcal{M}_{e_M} \subset \mathcal{M}_a$. Therefore $b \wedge c_M \neq 0$ and so $b \notin \bigcap c_M{}^{\perp}$.

QED

<u>Lemma 3.2.1</u> *Let A be a commutative semi-prime f-ring with 1 and bounded inversion. Then $S(A)^L = Q(S(A))$. If A is an f-algebra, then $S_{\mathbb{R}}(A)^L = Q(S_{\mathbb{R}}(A))$*

PROOF

Since $S(A)$ is an archimedean f-ring with bounded inversion, by Theorem 1.8 [24], $Q(S(A)) = S(A)^o$ the orthocompletion of $S(A)$. Since $S(A)$ is archimedean, $S(A)^o = S(A)^L$ (Theorem 8.2.5 [2]). The proof is identical if A is a vector lattice.

QED

It is in this context of lateral completions that the differences between $S(A)$ and $S_{\mathbb{R}}(A)$ first becomes apparent as the following example indicates.

<u>Example</u> Let A be the ring of eventually constant real valued sequences. Then A is an f-algebra. The idempotents of A consist of $\{0,1\}$ sequences that are eventually constant. Then $S(A)$ consists of the eventually constant rational sequences and $S(A)^L = \prod \mathbb{Q}$, whereas, $S_{\mathbb{R}}(A)$ consists of the eventually constant real sequences and $S_{\mathbb{R}}(A)^L = \prod \mathbb{R}$.

The first main result is the following theorem which characterizes specker-type f-rings in terms of the complete ring of quotients. An analogous result holds for f-algebras.

<u>Theorem 3.2.3</u> *Let A be a commutative semi-prime f-ring with 1 and bounded inversion. Then the following are equivalent.*

1. A is an specker-type ring.

2. A is a subring of $Q(S(A))$.

3. $S(A)^L = Q(A)$.

PROOF

That (1) implies (2) follows directly from the definition since $Q(S(A))$ is the maximal $S(A)$-essential extension of $S(A)$. If $S(A) \subset A \subset Q(S(A))$ then A is an $S(A)$-essential extension of $S(A)$ so that (2) implies (1). We will now show that (2) implies (3). Suppose that $A \subset Q(S(A))$. For $0 \neq a \in Q(S(A))$, by Theorem 3.1.1 there is a $0 \neq b \in S(A)$ such that $0 \neq ab \in S(A)$. Since $S(A) \subset A$, $Q(S(A))$ is an A-essential extension of A. By Lemma 3.1.1 and Theorem 3.1.8, $Q(A) \subset Q(Q(S(A))) = Q(S(A))$. Since A is a subring of $Q(S(A))$, A is an $S(A)$-essential extension of $S(A)$ so that $Q(S(A)) \subset Q(A)$. Therefore $Q(A) = Q(S(A))$. By Lemma 3.2.1, $S(A)^L = Q(A)$. Since A is a subring of $Q(A)$, that (3) implies (2) also follows directly from Lemma 3.2.1.

QED

We will now consider how specker-type rings, quasi-specker rings and local-global rings relate to one another. Recall that for any commutative semi-prime f-ring with identity and bounded inversion, $J(A) \subset \{x \in A : |x| << 1\}$, and therefore if A is archimedean then $J(A) = 0$.

<u>Corollary 3.2.1</u> *Let A be a commutative semi-prime f-ring with identity and bounded inversion. If A is an specker-type ring, then A is archimedean. The same conclusion obtains for f-algebras.*

PROOF

Since A is a specker-type ring, A is an f-subring of $S(A)^L$. Since $S(A)$ is archimedean, by Theorem 8.2.5 [2] its lateral completion $S(A)^L$ is as well. An f-subring of an archimedean f-ring is archimedean, so that A is archimedean.

QED

<u>Theorem 3.2.4</u> *Let A be a commutative semi-prime f-ring with identity and bounded inversion. If A is a specker-type ring then A is a quasi-specker ring. The same conclusion obtains for f-algebras.*

PROOF

Let $P \in \mathcal{P}(A)$. Since A is an f-ring, P is a ring ideal and therefore an $S(A)$ submodule of A. Since A is a specker-type ring, $P \cap S(A) \neq \emptyset$. Therefore, by Theorem 3.1.3, A is a quasi-specker ring.

QED

<u>Theorem 3.2.5</u> *Let A be an archimedean commutative semi-prime f-ring with identity and bounded inversion. If A is a local-global ring, then A is a quasi-specker ring.*

PROOF

Let $P \in \mathcal{P}(A)$ and let $0 \neq a \in P$. As in the proof of Theorem 3.2.2, there is an $e \in A$ idempotent with $e \in a^{\perp\perp}$. Since $a \in P$, $a^{\perp\perp} \subset P$, so that $e \in P \cap S(A)$ and A is a quasi-specker ring.

QED

We will eventually see examples in the case $A = C(X)$ to show that none of the two previous implications reverse.

3.3 Quasi-Specker Spaces and Specker Spaces

In this section we will apply the preceding results for the case $A = C(X)$. Following the exposition in *Rings of Continuous Functions* by Gillman and Jerison [15], we consider for a space X, the ring of continuous real valued functions $C(X)$. All spaces will be assumed to be Tychonoff. We recall the following notation. For $f \in C(X)$, $coz(f) = \{x \in X : f(x) \neq 0\}$ and $Z(f) = \{x \in X : f(x) = 0\}$. Some observations and recollections are in order.

1. For X Tychonoff, $\{coz(f) : f \in C(X)\}$ is a base for the open sets of X.
2. If we define a partial ordering on $C(X)$ by $f \geq 0$ if $f(x) \geq 0$ for all $x \in X$, and operations pointwise, then $C(X)$ is a commutative semi-prime f-ring with bounded inversion and identity.

We will use the following notational conventions. $Q(X) = Q(C(X))$, $q(X) = q(C(X))$, $S(X) = S_{\mathbb{R}}(C(X))$, $C^*(X) = \{f \in C(X) : f \leq n1 \text{ for some } n \in \mathbb{N}\}$.

<u>Lemma 3.3.1</u> *Let X be a Tychonoff space. Then $C(\beta X)$ is ℓ-isomorphic to $C^*(X)$ and $S(\beta X)$ is ℓ-isomorphic to $S(X)$.*

PROOF

That $C(\beta X)$ and $C^*(X)$ are isomorphic as rings is Theorem 4.6(i) in [27] and is given by $f \mapsto f^\beta$ from $C^*(X)$ to $C(\beta X)$. That this map preserves order follows from the density of X in βX. The second ℓ-isomorphism follows from the first and the observation that $S(C^*(X)) = S(X)$.

QED

In subsequent arguments involving $C(\beta X)$ and $S(\beta X)$, we will use the above ℓ-isomorphisms extensively; that is, to prove a result for $C(\beta X)$, we will prove the

result for $C^*(X)$. The next theorem is an improvement of Theorem 3.2.3 for the case $A = C(X)$

<u>Theorem 3.3.1</u> *Let X be a Tychonoff space. The following are equivalent.*

1. *X is a specker space.*
2. *For every $0 \neq f \in C(X)$ there is a $K \subset X$ clopen and a $0 \neq c \in \mathbb{R}$ such that $f|_K = c$.*
3. *βX is a specker space.*
4. *$S(X)^L = Q(X)$.*

PROOF

Suppose that X is a specker space. Let $0 \neq f \in C(X)$. Since X is a specker space, there is an $s \in S(X)$ such that $0 \neq sf \in S(X)$. We can write $s = r_1\chi_{K_1} + \cdots + r_n\chi_{K_n}$ where we may assume that the r_i are distinct, nonzero and the K_i are disjoint, clopen. Since $0 \neq sf \in S(X)$, we can write $sf = q_1\chi_{K_1'} + \cdots + q_m\chi_{K_m'}$ where again, the q_i are distinct, nonzero, and the K_i' are disjoint, clopen. Then for any $1 \leq i \leq m$, $sf|_{K_i'} = q_i \neq 0$. In particular, $s|_{K_i'} \neq 0$. Then $K_i' \subset coz(s)$ so that $K_i' \cap K_j \neq \emptyset$ for some $1 \leq j \leq n$. Let $T = K_i' \cap K_j$. Then T is clopen and $sf|_T = r_j f|_T = q_i$, so that $f|_T = q_i/r_j \neq 0$.

Suppose now that (2) holds. Let $0 \neq f \in C(X)$. Then there is a $\emptyset \neq K \subset X$ clopen and a $0 \neq c \in \mathbb{R}$ with $f|_K = c$. Let $s = \chi_K$. Then $s \in S(X)$ and $0 \neq sf \in S(X)$. Therefore X is a specker space.

We will next show that (2) and (3) are equivalent. Suppose that (2) holds. It suffices to show that $C^*(X)$ is a specker-type ring. Let $0 \neq f \in C^*(X)$. Since $C^*(X) \subset C(X)$, there is a $K \subset X$ clopen and a $0 \neq c \in \mathbb{R}$ such that $f|_K = c$. Let $s = \chi_K$. Then $s \in S(C^*(X)) = S(X)$ and $0 \neq sf \in S(X)$.

Now suppose that βX is a specker space. Let $0 \neq f \in C(X)$. We will first show that we may assume that $0 \leq f$. If $0 \not\leq f$ then either $f \leq 0$ or $coz(f \vee 0) \neq \emptyset$. If $f \leq 0$ then $-f \geq 0$ and $-f|_K = c \neq 0$ implies $f|_K = -c \neq 0$. If $coz(f \vee 0) \neq \emptyset$ then there is a $\emptyset \neq T \subset X$ with $T \subset coz(f \vee 0)$. Then $0 \leq g = f|_T \in C(X)$. If there is a $\emptyset \neq K \subset X$ clopen and a $c \neq 0$ with $g|_K = c$, then $g|_K = f|_{T \cap K} = c$ with $\emptyset \neq T \cap K$ clopen.

We will show that if $C^*(X)$ is a specker-type ring then (2) holds. If $f \leq 1$, then $f \in C^*(X)$ and we are done. Assume then that $f \not\leq 1$. Then $(f \vee 1) \geq 1$ and $(f \vee 1) \neq 1$. Since $C(X)$ has bounded inversion, $(f \vee 1)^{-1} \in C(X)$ and $0 \leq 1 - (f \vee 1)^{-1} \leq 1$ with $(f \vee 1)^{-1} \neq 0, 1$. In particular, $1 - (f \vee 1)^{-1} \in C^*(X)$. Then there is a $\emptyset \neq K \subset X$ clopen, and a $c \neq 0$ with $1 - (f \vee 1)^{-1}|_K = c$, where $0 < c < 1$. Now, $0 < (f \vee 1)^{-1}|_K = 1 - c < 1$ so that $(f \vee 1)|_K = \frac{1}{1-c} > 1$. Therefore, $f|_K > 1|_K$ so that $f|_K = (f \vee 1)|_K = \frac{1}{1-c} \neq 0$.

The equivalence of (1) and (4) follows directly from Theorem 3.2.3, taking $A = C(X)$.

QED

Since a subset of X is clopen if and only if its characteristic function is continuous, it is straightforward to show that if X is a (quasi-)specker space and $Y \subset X$ is clopen, then Y is a (quasi-)specker space. As immediate corollaries of Theorem 3.3.1, we have the following results.

<u>Corollary 3.3.1</u> *If X contains a dense set of isolated points then X is a specker space.*

PROOF

If $0 \neq f \in C(X)$, then by density, there is an isolated point x with $f(x) \neq 0$. Take $K = \{x\}$ and apply (2) in Theorem 3.3.1.

QED

Definition 3.3.1 *A point p in X is called a p-point if $p \in Z(f)$ implies that $p \in int(Z(f))$.*

Corollary 3.3.2 *If X is zero-dimensional and contains a dense set of p-points, then X is a specker space.*

PROOF

We will again appeal to (2) in Theorem 3.3.1. Suppose that $0 \neq f \in C(X)$. By density, there is a p-point p such that $f(p) = c \neq 0$. Since p is a p-point, there is an open set O with $p \in O$ such that $f|_O = c$. Since X is zero-dimensional, there is a K clopen with $p \in K \subset O$. Then $f|_K = c$ and therefore X is a specker space.

QED

Definition 3.3.2 *A collection B of open sets of X is called a π-base if for every non-empty open set O there is a $\emptyset \neq U \in B$ with $U \subset O$.*

We have the following near analogue of Theorem 3.3.1 for quasi-specker spaces.

Theorem 3.3.2 *Let X be a Tychonoff space.*

1. *X is a quasi-specker space.*

2. *βX is a quasi-specker space.*

3. *X has a clopen π-base.*

Then (1) and (2) are equivalent; (1) implies (3), and if X is compact then (3) implies (1).

PROOF

Suppose that X is a quasi-specker space. Then $S(X)$ is order large in $C(X)$. Since $S(X) \subset C^*(X) \subset C(X)$ as f-rings, $S(X)$ is order large in $C^*(X)$. Suppose now that

βX is a quasi-specker space. Let $0 \leq f \in C(X)$. Then $0 \leq (f \wedge 1) \in C^*(X)$, so there is an $s \in S(X)$ and an $n \in \mathbb{N}$ such that $0 \leq s \leq n(f \wedge 1)$. But $n(f \wedge 1) = nf \wedge n \leq nf$ so that X is a quasi-specker space.

To show that (1) implies (3), let $\emptyset \neq O \subset X$ be open. Since X is Tychonoff, there is an $f \in C(X)$ with $\emptyset \neq coz(f) \subset O$. By the hypothesis, there is an $0 \neq s \in S(X)$ and an $n \in \mathbb{N}$ such that $0 \leq s \leq nf$. Say $s = r_1\chi_{K_1} + \cdots + r_n\chi_{K_n}$ where as before the r_i's are distinct, nonzero and the K_i's are nonempty disjoint clopen subsets of X. Then for any i, $K_i \subset coz(nf) = coz(f) \subset O$. Therefore X has a clopen π-base.

Suppose now that X is compact with a clopen π-base. Let $0 \leq f \in C(X)$ with $0 \neq f$. Then there is a $\emptyset \neq K$ clopen with $K \subset coz(f)$. Since X is compact and K is clopen, K is compact. Therefore there is an $0 < \alpha \in \mathbb{R}$ such that $\alpha = min\{f(x) : x \in K\}$. Choose $n \in \mathbb{N}$ so that $n \geq \frac{1}{\alpha}$. If we take $s = \chi_K \in S(X)$, then $nf \geq \frac{1}{\alpha}f \geq s$ and X is a quasi-specker space.

QED

We have seen in the previous section that for f-rings (respectively f-algebras) if A is a specker-type ring (algebra) then A is a quasi-specker ring, and if A is a local-global ring with zero Jacobson radical, then A is a specker-type ring (algebra). In the present context these results translate as follows.

First, by taking $A = C(X)$ in Theorem 3.2.4, we get the following corollary.

<u>Corollary 3.3.3</u> *If X is a specker space then X is a quasi-specker space.*

<u>Corollary 3.3.4</u> *If X is strongly zero-dimensional then X is a quasi-specker space.*

PROOF

If X is strongly zero-dimensional then βX is compact and zero-dimensional. $C(\beta X)$ is then local-global, and as $C(Y)$ is archimedean for any space Y, βX is

a quasi-specker space, by Theorem 3.2.5. Therefore, by Theorem 3.3.2, X is a quasi-specker space.

QED

The following examples show that the converse of the above corollaries do not hold and that being a specker space is independent of strong zero-dimensionality.

Example Of a space X which is a specker space and hence a quasi-specker space, but which is not strongly zero-dimensional.

Let $\mathbb{N}$ be the natural numbers with the discrete topology and let $\alpha\mathbb{N}$ denote the one point compactification of $\mathbb{N}$ where we denote by ∞ the point adjoined. Let $X = [0, 1] \times \alpha\mathbb{N}$ where the topology is a refinement of the product topology such that points of the form $\{(r, n)\}$ for $n \in \mathbb{N}$ are isolated.

The idea is that the isolated points of $\alpha\mathbb{N}$ remain isolated. Suppose now that $0 \neq f \in C(X)$. Then $f(r, n) = c \neq 0$ for some $(r, n) \in X \setminus \{[0, 1] \times \{\infty\}\}$. Since (r, n) is isolated, $s = \chi_{\{(r,n)\}} \in S(X)$ and $0 \neq c\chi_{\{(r,n)\}} = sf \in S(X)$. Therefore X is a specker space. Since X contains $[0, 1]$ as a connected subset, it is not (strongly) zero-dimensional.

Example Of a space X which is strongly zero-dimensional and hence a quasi-specker space, but is not a specker space.

Let $X = \mathbb{Q}$ as a subspace of $\mathbb{R}$ with the topology generated by the open intervals. Then X is strongly zero-dimensional; intervals of the form (r, s) with $r, s \in \mathbb{R} \setminus \mathbb{Q}$ are a clopen base. Let $f \in C(X)$ be the identity map. Suppose there is an $s \in S(X)$ such that $0 \neq sf \in S(X)$. Then write $sf = r_1\chi_{K_1} + \cdots + r_n\chi_{K_n}$ where the r_i's are distinct nonzero and the K_i's are nonempty disjoint clopen and $s = q_1\chi_{K_1'} + \cdots + q_m\chi_{K_m'}$ similarly. Then there are i, j such that $T = K_i \cap K_j' \neq \emptyset$. T is clopen and $f|_T = r_i/q_j$. This is a contradiction since points are not clopen.

These examples show that the following hold.

1. That "X is a specker space" is independent of whether or not X is strongly zero-dimensional.

2. That X is a quasi-specker space does not imply that X is a specker space.

3. That X is a quasi-specker space does not imply that X is strongly zero-dimensional.

3.4 The Absolute of a Hausdorff Space

In what follows we will consider one construction of the absolute of a Hausdorff space and some of its properties. The following is the construction presented in "Extensions and Absolutes of Hausdorff Spaces" [27], Chapter 6.6 of the Gleason Absolute of a compact Hausdorff space first done by A.M. Gleason [17].

We will need the following definitions.

Definition 3.4.1 *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then f is said to be perfect if f is closed and for each $y \in Y$, $f^{-1}(y)$ is a compact subset of X.*

In the definition of perfect maps there is no assumption about continuity, but in the case that f is continuous and X is compact, f is perfect.

Definition 3.4.2 *Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then f is said to be irreducible if f is closed, onto, and if K is a proper closed subset of Y, then $f(K) \neq Y$.*

Irreducible maps have many interesting and useful properties. For more on irreducible maps, see Chapter 6.5 [27]. In particular, irreducible maps have the following properties.

<u>Lemma 3.4.1</u> *Let X, Y, and Z be spaces, $f : X \to Y$ an irreducible map. Then,*

1. *If $g : Y \to Z$ is an irreducible map then so is the composite $g \circ f : X \to Z$.*

2. *If $\emptyset \neq U \subset X$ is open, then $int(f(U)) \neq \emptyset$.*

3. *If $W \subset Y$ is dense, then $f^{-1}(W)$ is dense and $f|_{f^{-1}(W)} : f^{-1}(W) \to W$ is irreducible.*

<u>Example</u> Let $\alpha\mathbb{N}$ denote the one point compactification of the naturals. The map from $\phi : \beta\mathbb{N} \to \alpha\mathbb{N}$ given by $\phi(n) = n$ for $n \in \mathbb{N}$ and $\phi(\beta\mathbb{N} \setminus \mathbb{N}) = \infty$ is irreducible. Moreover, if $\overline{\mathbb{N}}$ is any compactification of $\mathbb{N}$ then the Stone extension $i^{\beta} : \beta\mathbb{N} \to \overline{\mathbb{N}}$ of the embedding of $\mathbb{N}$ in $\overline{\mathbb{N}}$ is irreducible.

<u>Definition 3.4.3</u> *A space X is said to be extremally disconnected if the closure of each open set is open.*

If we restrict our attention to Tychonoff spaces we have the following, which is a compilation of results from [27], §6.2 Theorems (b) and (c), and [15] problem 3N4.

<u>Theorem 3.4.1</u> *Let X be a Tychonoff space. The following are equivalent.*

1. *X is extremally disconnected.*

2. *Disjoint open sets in X have disjoint closures.*

3. *Each dense (open) subset of X is extremally disconnected.*

4. *Each dense (open) subset of X is C^* embedded.*

5. *βX is extremally disconnected.*

6. *$C(X)$ is Dedekind complete.*

Example Clearly any discrete space D is extremally disconnected. Theorem 3.4.1 tells us then that βD is also extremally disconnected. Both D and βD contain a dense set of isolated points. To find an extremally disconnected space without isolated points, begin with a complete atomless boolean algebra $\mathcal{B}$. Then $St(\mathcal{B})$ is a compact extremally disconnected space without isolated points.

Definition 3.4.4 *For a space X, a subset K of X is called a regular closed subset if $K = cl(int(K))$*

Now let X be a compact Hausdorff space and let $\mathcal{R}(X)$ denote the regular closed subsets of X. We define the following operations on $\mathcal{R}(X)$.

Definition 3.4.5 *For $A, B \in \mathcal{R}(X)$, $\mathcal{B} \subset \mathcal{R}(X)$, define*

1. *A binary meet by $A \wedge B = cl(int(A \cap B))$.*
2. *A binary join by $A \vee B = A \cup B$.*
3. *Complementation by $A' = X \setminus int(A)$.*

and an infinitary meet and join by

4. $\wedge\mathcal{B} = cl(int(\cap\mathcal{B}))$ *and* $\vee\mathcal{B} = cl(int(\cup\mathcal{B}))$

By Proposition 2.2(c) and Example 3.1(e)(4) in [27], $(\mathcal{R}(X), \wedge, \vee, ')$ is a complete Boolean algebra. Let EX denote the Stone dual of $\mathcal{R}(X)$. Then EX is a compact, Hausdorff extremally disconnected space. For emphasis we will recall here that the points of EX are $\mathcal{R}(X)$-ultrafilters and that the basic open sets of EX are of the form: for $K \in \mathcal{R}(X)$, $O_K = \{\alpha \in EX : K \notin \alpha\}$. By Theorem 6.6(e) in [27] the map $e : EX \to X$, defined by $e(\alpha) = \cap\alpha$, is a continuous irreducible surjection.

The above construction generalizes to the non-compact case by taking EX to be the subspace of the Stone dual of $\mathcal{R}(X)$ whose points are the fixed ultrafilters. In this case, EX is an extremally disconnected zero-dimensional dense subspace of the Stone dual of $\mathcal{R}(X)$ and $e : EX \to X$ as defined above is a continuous perfect irreducible surjection. We will call e the *canonical surjection.*

The space EX as defined above is called the *Gleason Absolute* of X, in the case that X is compact, and the *Illiadis Absolute* for X non-compact. We will agree to refer to EX as the *absolute* of X in either case. In addition to the above mentioned properties of the absolute, EX is unique in the following sense.

Theorem 3.4.2 *Let X be a Hausdorff space and let (Y, ψ) be such that Y is a zero-dimensional, extremally disconnected space and $\psi : Y \to X$ is a perfect irreducible function. Then there is a homeomorphism $\phi : EX \to Y$ such that $\psi \circ \phi = e$.*

Of particular interest will be the relationship between o-large embeddings of f-rings and irreducible maps of topological spaces. Recall that if A is a commutative semi-prime f-ring with identity and bounded inversion and if B is an f-subring of A with bounded inversion and $1 \in B$, then there is an induced map $\phi : Max(A) \to Max(B)$ that is a continuous surjection.

We translate part of the exposition contained in Chapter 11 of "Rings of Quotients of Rings of Functions" [12] into a language suitable for our purposes. The discussion that follows was originally done for annihilator ideals of semi-prime rings and the Boolean algebra of regular open sets. Let $P \in \mathcal{P}(A)$ and consider $K_P = \{M \in Max(A) : M \supset P\}$ and $O_P = \{M \in Max(A) : M \not\supset P^\perp\}$. Then K_P is closed, O_P is open and since maximal ideals are prime, $O_P \subset K_P$. It is always the case that $cl(O_P) = K_P$ and if $J(A) = 0$, then $int(K_P) = O_P$. In particular if $J(A) = 0$, then K_P is a regular closed set. Moreover, when $J(A) = 0$ the map $P \mapsto K_P$ is a

Boolean isomorphism from $\mathcal{P}(A)$ onto $\mathcal{R}(Max(A))$ (Theorem 11.8 [12]). Therefore, if $J(A) = 0$, the regular closed sets of $Max(A)$ are precisely those of the form $K_P = \{M \in Max(A) : M \supset P\}$ for some $P \in \mathcal{P}(A)$.

Theorem 3.4.3 *Let A, B and ϕ be as in the preceding paragraphs. If B is o-large in A, then ϕ is irreducible. If in addition A is archimedean then the converse holds.*

PROOF

Since $Max(A)$ is compact and $Max(B)$ is Hausdorff, ϕ is closed. Recall that ϕ is defined by $\phi(M) =$ the unique maximal ideal of B containing $M \cap B$. Denote by $\mathcal{N}_a$ the basic open sets of $Max(B)$ and by $\mathcal{M}_a$ the basic open sets of $Max(A)$. Let K be a proper closed subset of $Max(A)$. Then there is an $0 < a \in A$ such that $\mathcal{M}_a \subset Max(A) \setminus K$. Since B is o-large, there is a $b \in B$ and a $n \in \mathbb{N}$ such that $0 < b \leq na$. Then $\mathcal{M}_b \subset \mathcal{M}_a$ so that $K \subset Max(A) \setminus \mathcal{M}_b$. If $M \in Max(A) \setminus \mathcal{M}_b$, then $b \in M$ and as $b \in B$, $b \in M \cap Max(B) \subset \phi(M)$. Then $\phi(M) \in Max(B) \setminus \mathcal{N}_b$ and consequently $\phi(K) \subset Max(B) \setminus \mathcal{N}_b \neq Max(B)$. Therefore ϕ is irreducible.

For the converse we will show a bit more. That if $J(A) = 0$ and ϕ is irreducible then the contraction $P \mapsto P \cap B$ is a Boolean isomorphism from $\mathcal{P}(A)$ onto $\mathcal{P}(B)$. Then by Theorem 3.1.3, in the case that A is archimedean, the result follows. Also by Theorem 3.1.3, suffices to show that if $0 \neq a \in A$ then $a^{\perp\perp} \cap B \neq 0$. Suppose we know that this holds for $B(1) \subset A(1)$. Let $*$ denote the polar operation in $A(1)$. Let $0 \neq a \in A$. Since $A(1)$ is order large in A we have that $a^{\perp\perp} \cap A(1) \neq 0$. In fact, $a^{\perp\perp} \cap A(1) = (a \wedge 1)^{**}$. Therefore $(a \wedge 1)^{**} \cap B(1) \neq 0$, so that $a^{\perp\perp} \cap B \neq 0$. Since $Max(A) \cong Max(A(1))$, $Max(B) \cong Max(B(1))$ and $J(A) = 0$ implies $J(A(1)) = 0$, we may assume without loss of generality that A has strong unit.

Suppose that A has a strong unit with $J(A) = 0$, B is an f-subring of A and $\phi : Max(A) \to Max(B)$ is irreducible. Since A has strong unit, $\phi(M) = M \cap B$.

Let $0 \neq a \in A$. Since $J(A) = 0$, there is an $M \in Max(A)$ with $a \notin M$, so that $K = \{M \in Max(A) : M \supset a^{\perp\perp}\}$ is a proper regular closed subset of $Max(A)$. Since ϕ is irreducible $\phi(K)$ is a proper regular closed subset of $Max(B)$. Therefore there is a $Q \in \mathcal{P}(B)$ such that $\phi(K) = \{N \in Max(B) : N \supset Q\}$. Each $N \in \phi(K)$ is of the form $N = M \cap B$ for some $M \in K$ so that $\phi(K) = \{M \cap B : M \cap B \supset Q\}$.

Suppose now that $a^{\perp\perp} \cap B = 0$ and $Q \neq 0$. Then there is a $0 < q \in Q$ with $q \notin a^{\perp\perp}$. Since $q \notin a^{\perp\perp}$, there is a $0 < c \in a^{\perp}$ with $0 < cq$. Since $J(A) = 0$, there is an $M \in Max(A)$ with $cq \notin M$. Then $c \notin M$ and $q \notin M$. Since $c \notin M$, $a^{\perp} \not\subset M$ and as M is prime, $a^{\perp\perp} \subset M$. Then $M \in K$ so that $\phi(M) = M \cap B \supset Q$. This is a contradiction as $q \notin M$. Therefore, if $a^{\perp\perp} \cap B = 0$ then $Q = 0$. If $Q = 0$ then $\phi(K) = Max(B)$, but ϕ is irreducible and K is a proper closed subset of $Max(A)$ so that $Q \neq 0$, and hence $a^{\perp\perp} \cap B \neq 0$.

We have then for $0 \neq a \in A$ that $a^{\perp\perp} \cap B \neq 0$ and so by Theorem 3.1.3, the map $P \mapsto P \cap B$ is a Boolean isomorphism from $\mathcal{P}(A)$ onto $\mathcal{P}(B)$.

QED

We can also argue from A archimedean to the order largeness directly by noting that if A is archimedean and ϕ is irreducible, by Lemma 11.8 in [12], the map $P \mapsto \{M \in Max(A) : M \supset P\}$ is a Boolean isomorphism from $\mathcal{P}(A)$ onto $\mathcal{R}(Max(A))$. Since B also is archimedean we have that $\mathcal{P}(B)$ is Boolean isomorphic to $\mathcal{R}(Max(B))$. By the embedding theorem for archimedean l-groups, $B^e = A^e$. Then $B \subset A \subset B^e$ and since B is o-large in B^e, it is o-large in A.

3.5 Specker Spaces and Absolutes

In this section we look at the "inheritability" of being a specker-type space in the context of Tychonoff spaces vis-a-vis their absolutes.

<u>*Lemma 3.5.1*</u> *Suppose that $A = S_{\mathbb{R}}(A)$ and $A^L = A^e$. Then $Max(A)$ is a specker space.*

PROOF

Since $A = S_{\mathbb{R}}(A)$, A is hyper-archimedean and therefore, $Max(A) \cong Min(A)$ is zero-dimensional. Since $Max(A)$ is compact, by Corollary 2.3.4, $C(Max(A))$ is local-global. Then by Theorem 3.2.5, $C(Max(A))$ is a quasi-specker ring. That is $S(Max(A))$ is o-large in $C(Max(A))$, so that $C(Max(A))$ is an f-subring of $S(Max(A))^e$.

For any ring R, let $Id(R)$ denote the idempotents of R. For $a \in Id(A)$, the map $a \mapsto \chi_{\mathcal{M}_a}$ extends linearly to an f-ring isomorphism from A onto $S(Max(A))$. Therefore $S(Max(A))^e \cong A^e = A^L \cong S(Max(A))^L$. By Lemma 3.2.1, $S(Max(A))^L = Q(S(Max(A)))$ therefore $C(Max(A)) \subset Q(S(Max(A)))$.

QED

<u>*Lemma 3.5.2*</u> *If X is a quasi-specker space and $S(X)^L$ is o-essentially closed then X is a specker space.*

PROOF

Since $S(X)$ is a quasi-specker, $S(X)$ is o-large in $C(X)$. By Theorems 3.1.3 and 3.1.5, $S(X)^e = C(X)^e$. By Lemma 3.2.1, $S(X)^L = Q(S(X))$, therefore $C(X) \subset Q(S(X))$ and X is a specker space.

QED

In the preceding lemma we cannot omit the assumption that X is a quasi-specker space, as the following example shows.

<u>Example</u> Let $X = [0,1]$ with the open interval topology. Since X is connected, the only idempotents in $C(X)$ are the constant functions 0 and 1. Then $S(X) \cong \mathbb{R}$

and is essentially closed, as is $S(X)^L$, but X has no proper clopen subsets and is therefore not a specker space.

<u>Lemma 3.5.3</u> *If X is a compact quasi-specker space and EX is a specker space, then X is a specker space.*

PROOF

Let $0 \neq f \in C(X)$. Then $0 \neq f \circ e \in C(EX)$, where e is the canonical surjection. Since EX is a specker space, there is a nonempty clopen $K \subset EX$ and a $c \neq 0$ with $f \circ e(K) = c$. Since K is clopen hence regular closed and e is irreducible, $e(K)$ is a nonempty regular closed subset of X. In particular, $int(e(K)) \neq \emptyset$. Since X is compact and quasi-specker, there is a nonempty $B \subset X$ clopen with $B \subset int(e(K))$. Then $f(B) = c$ so that X is a specker space.

QED

The preceding result generalizes to the following.

<u>Theorem 3.5.1</u> *If X is a quasi-specker space and EX is a specker space, then X is a specker space.*

PROOF

Since X is a quasi-specker space, βX is a compact quasi-specker space. Since EX is a specker space, $\beta(EX)$ is a specker space. Since X is Tychonoff, by Theorem 6.9(b)(4) in [27], $\beta(EX) \cong E(\beta X)$ so that $E(\beta X)$ is a specker space. By Lemma 3.5.3, βX is a specker space and therefore X is a specker space.

QED

For extremally disconnected spaces the following characterization obtains.

<u>Theorem 3.5.2</u> *If X is extremally disconnected then X is a specker space if and only if $S(X)^L$ is essentially closed.*

PROOF

Suppose that X is a specker space. Since X is extremally disconnected, by Theorem 3.4.1, $C(X)$ is Dedekind complete. Since X is a specker space, $C(X)$ is an o-large subring of $S(X)^L$. Since X is Tychonoff and extremally disconnected, X is strongly zero-dimensional and hence a quasi-specker space. In particular, $S(X)$ is o-large in $C(X)$. Since $C(X)$ is Dedekind complete, $S(X)^\wedge$ is an o-large subring of $C(X)$ and hence of $S(X)^L$. But then $S(X)^e = (S(X)^\wedge)^L \subset S(X)^L \subset S(X)^e$ and $S(X)^L$ is essentially closed.

Suppose now that $S(X)^L$ is essentially closed. Since X is extremally disconnected, X is a quasi-specker space and the result follows from Lemma 3.5.2.

QED

We have shown that for quasi-specker spaces, if the absolute is a specker space, then the space itself is a specker space. We would like to reverse this implication. So far we have only partial success.

<u>*Theorem 3.5.3*</u> *Let X be a quasi-specker space. If $S(X)^L$ is essentially closed then EX is a specker space.*

PROOF

The continuous map $e : EX \rightarrow X$ induces an f-ring homomorphism $C(e) : C(X) \rightarrow C(EX)$ given by; for $f \in C(X)$, $C(e)(f) = f \circ e$. Since e is onto, $C(e)$ is an embedding and since e is perfect and irreducible, this embedding is o-large. Then since X is a quasi-specker space, $S(X)$ is o-large in $C(X)$ and therefore $S(X)$ is o-large in $C(EX)$. The restriction of $C(e)$ to $S(X)$ is an embedding into $S(EX)$, and as $S(X)$ is o-large in $C(EX)$, $S(X)$ is o-large in $S(EX)$. Therefore,

$S(X)^L \subset S(EX)^L$. By the hypothesis, $S(X)^L = S(X)^e$. Since $S(X)$ is o-large in $C(EX)$, $C(EX) \subset S(X)^e$. Thus $C(EX) \subset S(EX)^L$ and EX is a specker space.

QED

Definition 3.5.1 *A space X is said to have the countable chain condition if every collection of disjoint open sets is at most countable.*

The following conjecture is the motivation for much of what follows.

Conjecture 3.5.1 *Let X be a compact Tychonoff space with the countable chain condition. If X is a specker space, then EX is a specker space.*

Recall that a collection of open subsets B of a space X is called a *π-base* for X if for every nonempty open $O \subset X$ there is a nonempty $U \in B$ with $U \subset O$. As a special case of this conjecture we will consider the case for X a compact Tychonoff space with a countable π-base where we can get a positive result. We will then look at what progress has been made with regards to the conjecture itself. We need the following notion which is due to J. Martinez [23].

Definition 3.5.2 *A Tychonoff space X is said to be a weakly specker space if whenever $0 \neq f \in C(X)$, then for each open set V on which f does not vanish, there exists an open $U \subset V$ such that $f|_U$ is nonzero and constant.*

Lemma 3.5.4 *If X is weakly specker, then for any $0 \neq f \in C(X)$, there is a collection of open sets with dense union such that f is constant on each open set.*

PROOF

Let $S = \{O_\alpha : f|_{O_\alpha} = c_\alpha \neq 0\} \cup \{O_\alpha : f|_{O_\alpha} = 0\}$. If S is not dense, then $V = X \setminus cl(S)$ is a nonempty open set on which f does not vanish. Since X is

specker, there is an open $U \subset V$ such that $f|_U$ is nonzero and constant. This is a contradiction. Therefore, S is dense.

QED

The following is an unpublished result due to W. W. Comfort.

Theorem 3.5.4 *If X is a metric space with no isolated points, then there is a continuous $f : X \longrightarrow [0,1]$ such that if U is a nonempty open subset of X, then $|f(U)| \geq 2$.*

For our purposes, the importance of Comfort's Theorem is the following corollary.

Corollary 3.5.1 *If X is a weakly specker metric space, then X contains a dense set of isolated points.*

PROOF

If X contains no isolated points, then the function guranteed by Theorem 3.5.4 is not constant on any open set. Therefore a weakly specker metric space contains at least one isolated point. Let D be the set of isolated points of X. If $cl(D) \neq X$, then $X \backslash cl(D)$ is a nonempty open set without isolated points. Therefore $K = cl(X \backslash cl(D))$ is a metric space that is a closed subspace of X that contains no isolated points. Let $0 \neq f \in C^*(K)$ be the function guaranteed by Theorem 3.5.4. By Tietze's extension theorem, there is an $f \in C(X)$ such that $f|_K = g$. Let $\{O_\alpha\}$ be a collection of open sets such that $f|_\alpha$ is constant. Since $\bigcup\{O_\alpha\}$ is dense, there is an α such that $X \setminus cl(D) \cap O_\alpha = U \neq \emptyset$. Then U is an open subset of K such that $|f(U)| = 1$. This is a contradiction. Therefore $cl(D) = X$ and hence D is dense.

QED

Suppose now that X is a specker space and that $\theta : X \longrightarrow Y$ is a continuous perfect irreducible map. Let $g \in C(Y)$. Then $g \circ \theta \in C(X)$. Since X is a specker space, there is a $K \subset X$ clopen and a $0 \neq c \in \mathbb{R}$ such that $g \circ \theta(K) = c$. But θ is irreducible so that $int(\theta(K)) \neq \emptyset$ is open. This proves the following lemma.

<u>Lemma 3.5.5</u> *If X is a specker space and $\theta : X \to Y$ is a continuous perfect irreducible map, then Y is a weakly specker space.*

For X, a compact Tychonoff space with a countable π-base, we will construct a quotient space that is an irreducible image of X and is also a metric space. This is the content of the following theorem.

<u>Theorem 3.5.5</u> *If X is a compact Tychonoff space with a countable clopen π-base, then there is a metric space Y and an irreducible surjection $p : X \to Y$. In particular, $EX \cong EY$.*

PROOF

Let $\Pi' = \{B_n : n \in \mathbb{N}\}$ be a countable clopen π-base for X. Let Π be the Boolean algebra generated by Π'. Then Π is a countable clopen π-base for X. Define a relation on X by $x \sim y$ if x and y are in exactly the same sets of Π. It is easily verified that this is an equivalence relation on X. Let Y be the quotient space of X modulo this equivalence relation and let $p : X \to Y$ be the projection map. Since p is continuous and X is compact, Y is compact.

Let $[x]$ denote the equivalence class of x in X. We claim that $[x] = \cap\{B \in \Pi : x \in B\}$. For if $y \in [x]$ then $y \in B$ for every B with $x \in B$ so that $y \in \cap\{B \in \Pi : x \in B\}$. Conversely, if $y \in \cap\{B \in \Pi : x \in B\}$ then $x \in B$ implies $y \in B$. Since Π is a Boolean algebra, if there is a $B \in \Pi$ with $y \in B$ and $x \notin B$, then $X \setminus B \in \Pi$ has $y \notin X \setminus B \supset \cap\{B \in \Pi : x \in B\}$. This is a contradiction. Thus $y \in B$ implies $x \in B$ so that $y \in [x]$. In particular, since each $B \in \Pi$ is clopen any such intersection is closed so that the points of Y are closed.

We will next show that the projection map is closed. By Proposition 1.6, Chapter VI [10], it suffices to show that if $K \subset X$ is closed, then $p^{-1}(p(K))$ is closed. Let

$K \subset X$ be closed, and let $x \in X \setminus p^{-1}(p(K))$. Then $p(x) \notin p(K)$, and since Π is a Boolean algebra, there is a $B \in \Pi$ with $x \in B$. Therefore, for every $z \in K$ there is a $B_z \in \Pi$ with $x \in B_z$ and $z \notin B_z$. Then $\{X \setminus B_z : z \in K\}$ is an open cover of K. Since K is compact, there is a finite subcover say $\{X \setminus B_i : 1 \leq i \leq n\}$. We then have that $x \in \bigcap_{i=1}^{n} B_i \subset X \setminus K$ and since each B_i is clopen, $\bigcap_{i=1}^{n} B_i$ is clopen. It remains to show that $\bigcap_{i=1}^{n} B_i \subset X \setminus p^{-1}(p(K))$. Suppose that $z \in (\bigcap_{i=1}^{n} B_i) \cap p^{-1}(p(K))$. Then $p(z) = p(y)$ for some $y \in K$. But then $z \in \bigcap_{i=1}^{n} B_i$ implies $y \in \bigcap_{i=1}^{n} B_i \subset X \setminus K$; a contradiction. Therefore $\bigcap_{i=1}^{n} B_i \subset X \setminus p^{-1}(p(K))$ and $X \setminus p^{-1}(p(K))$ is open, so that $p^{-1}(p(K))$ is closed.

Now let K be a proper closed subset of X and suppose that $p(K) = Y$. Since $\emptyset \neq X \setminus K$ is open and Π is a π-base, there is a $B \in \Pi$ with $B \subset X \setminus K$. Let $x \in B$. Since $p(K) = Y$, there is a $y \in K$ with $p(y) = p(x)$, but this implies that $y \in B$. A contradiction. Therefore p is irreducible.

We now have that p is a continuous irreducible map from a compact Tychonoff space X onto a compact T_1 space Y. Then p is perfect and by Theorem 5.2, Chapter XI [10], Y is regular.

We will next show that Y is metrizable by showing that Y is second countable and appealing to a result due to P.Urysohn [10] which says for second countable spaces, regularity is equivalent to metrizability.

Let $y \in Y$. Then $y = p(x)$ for some $x \in X$. For every $B \in \Pi$ with $x \in B$, $y = p(x) \in p(B)$ so that $y \in \bigcap\{p(B) : B \in \Pi \text{ and } x \in B\}$. For $B \in \Pi$, $B \subset p^{-1}(p(B))$. If $x \in p^{-1}(p(B))$, then $p(x) \in p(B)$ so that $p(x) = p(z)$ for some $z \in B$. But then $x \in B$ and therefore $B = p^{-1}(p(B))$. Since B is open and the open sets U of Y are precisely those for which $p^{-1}(U)$ is open, we have that $p(B)$ is open for every $B \in \Pi$. Since Π is countable, $\{p(B) : B \in \Pi \text{ and } x \in B\}$ is at most countable. Let $y' \in Y$ with $y' \neq y$. Then $y' = p(x')$ for some $x' \in X$. Since $y' \neq y$, there is a $B \in \Pi$ with $x \in B$,

$x' \notin B$. If $p(x') \in p(B)$ then there is a $z \in B$ with $p(x') = p(z)$ and then $x' \in B$. Therefore, $y' = p(x') \notin p(B)$ and as $x \in B$, $p(B) \supset \bigcap\{p(B) : B \in \Pi \text{ and } x \in B\}$. Therefore $\{y\} = \bigcap\{p(B) : B \in \Pi \text{ and } x \in B\}$ so that every point of Y is a G_δ. In particular, by Theorem 3.5(l)(1) [27], since Y is compact and every point is a G_δ, Y is first countable.

Now let $\emptyset \neq U \subset Y$ be open and let $y \in U$. By the above, there is a countable subset of Π say $\{B_i : i \in \mathbb{N}\}$ such that $\{y\} = \bigcap\{p(B_i) : i \in \mathbb{N}\}$. We have already seen that each $p(B)$ is open for $B \in \Pi$ and, since p is a closed map, each $p(B)$ is clopen. Then $\{Y \setminus p(B_i) : i \in \mathbb{N}\}$ is an open cover of $Y \setminus U$. Since $Y \setminus U$ is closed and hence compact, there is a finite subset, $\{Y \setminus p(B_i) : 1 \leq i \leq n\}$ suitably reindexed, with $Y \setminus U \subset \bigcup\{Y \setminus p(B_i) : 1 \leq i \leq n\}$. Then $y \in \bigcap\{p(B_i) : 1 \leq i \leq n\} \subset U$ and $\bigcap\{p(B_i) : 1 \leq i \leq n\}$ is open. Let $B_0 = \bigcap\{B_i : 1 \leq i \leq n\}$. Clearly $B_0 \in \Pi$ and $p(B_0) \subset \bigcap\{p(B_i) : 1 \leq i \leq n\}$. If $z \in \bigcap\{p(B_i) : 1 \leq i \leq n\}$, then $p^{-1}(z) \subset p^{-1}(\bigcap\{p(B_i) : 1 \leq i \leq n\}) = \bigcap\{p^{-1}(p(B_i)) : 1 \leq i \leq n\} = \bigcap\{B_i : 1 \leq i \leq n\} = B_0$, so that $z \in B_0$. Therefore $p(B_0) = \bigcap\{p(B_i) : 1 \leq i \leq n\}$. This gives us that $\{p(B) : B \in \Pi\}$ is a countable basis for Y and therefore that Y is second countable. We now have a perfect irreducible map p from X onto a metric space Y. Then $p \circ \varepsilon : EX \to Y$ is a perfect irreducible map from an extremally disconnected zero-dimensional space onto Y. By the "uniqueness" of Theorem 3.4.2, $EX \cong EY$.

QED

As a corollary we have a partial converse of Theorem 3.5.1.

<u>Theorem 3.5.6</u> *Let X be a compact Tychonoff space with a countable π-base. If X is a specker space, then EX is a specker space. If this is the case, both X and EX contain a countable dense set of isolated points.*

PROOF

Let Π be a countable π-base for X and let $U \in \Pi$. Since X is Tychonoff, there is an $f \in C(X)$ with $coz(f) \subset U$. Since X is a specker space, there is a B clopen and a $0 \neq c \in \mathbb{R}$ such that $f(B) = c$. In particular, $B \subset coz(f) \subset U$. If we choose one such B for each $U \in \Pi$, then the resulting collection is a countable clopen π-base for X. By Theorem 3.5.5, there is a metric space Y and an irreducible map $p : X \to Y$. By Lemma 3.5.5, Y is a weakly specker metric space. By Corollary 3.5.1, Y contains a dense set of isolated points. Let D be this dense set of isolated points and let $\iota : EX \to Y$ be the irreducible map $p \circ e$ as in the proof of Theorem 3.5.5. By Lemma 3.4.1, $\iota^{-1}(D)$ is dense in EX. By Proposition 6.9(e) [27], the points of $\iota^{-1}(D)$ are all isolated. Therefore, EX contains a dense set of isolated points, and so EX is a specker space. Let $I(EX)$ and $I(X)$ denote the set of isolared points of EX and X respectively. Again, by Proposition 6.9(e) [27], $e|_{I(EX)}$ is a bijection from $I(EX)$ onto $I(X)$. If Π is a π-base for X, then Π must contain $I(X)$. Therefore if Π is countable, then $I(X)$ and hence $I(EX)$ are countable.

QED

Recall that a space is *locally compact* if each point has a compact neighborhood. In Chapter 6 [15], it is shown that X is open in βX if and only if X is locally compact. This gives most of the following corollary.

<u>Corollary 3.5.2</u> *Let X be a locally compact Tychonoff space with a countable π-base. If X is a specker space, then EX is a specker space.*

PROOF

Let Π be a countable π-base for X. Let $\emptyset \neq O \subset \beta X$ be open. Since X is locally compact, by the above observation, and since X is dense in βX, $O \cap X$ is a nonempty open subset of X. Then there is a $B \in \Pi$ with $B \subset O \cap X$. Again, as X is open in

βX, B is open in βX so that Π is a countable π-base for βX. Then βX is a compact specker space with a countable π-base so that by Theorem 3.5.6, $E(\beta X)$ is a specker space. Since $E(\beta X) \cong \beta(EX)$, EX is a specker space.

QED

To return to the standing of Conjecture 3.5.1, we begin with the following result which will allow us to restrict our attention to compact zero-dimensional spaces.

<u>Lemma 3.5.6</u> *If X is a compact quasi-specker space then there is a compact zero-dimensional space Y such that $EX \cong EY$.*

PROOF

Since X is a quasi-specker space, $S(X)$ is o-large in $C(X)$. By Theorem 3.4.3, the map $\phi : Max(C(X)) \to Max(S(X))$ isdn irreducible. Since X is compact, $X \cong Max(C(X))$ and since $S(X)$ is hyper-archimedean, $Max(S(X))$ is zero-dimensional.

QED

Now if X has the countable chain condition and $\theta : X \to Y$ is any continuous map, then Y has then countable chain condition. Therefore if X is a compact quasi-specker space with the countable chain condition, then X is co-absolute with a compact zero-dimensional space Y with the countable chain condition. The advantage of dealing with compact zero-dimensional spaces is that in the context of the following theorem, due to A. Hager, we can translate our topological problem to one of Boolean algebras.

<u>Definition 3.5.3</u> *Let $\mathcal{A}$ be a Boolean algebra. For $\mathcal{C}, \mathcal{D} \subset \mathcal{A}$ we say that $\mathcal{C}$ refines $\mathcal{D}$, denoted by $\mathcal{C} \prec \mathcal{D}$, if for every $c \in \mathcal{C}$ there is a $d \in \mathcal{D}$ such that $c \leq d$.*

<u>Definition 3.5.4</u> *Let $\mathcal{A}$ be a Boolean algebra. A* quasi-cover *of $\mathcal{A}$ is $0 \notin \mathcal{C} \subset \mathcal{A}$ with $\bigvee \mathcal{C} = 1$. A* cover *is a finite quasi-cover. A* partition *is a cover by pairwise disjoint elements.*

<u>*Definition 3.5.5*</u> *A Boolean algebra is called specker if every sequence of covers has a common refinement by a quasi-cover.*

We will need the following lemma.

<u>*Lemma 3.5.7*</u> *A zero-dimensional space X is a specker space if and only if for every $0 \neq f \in C(X)$ there is a $\mathcal{C} \subset \mathcal{B}(X)$ such that $\bigcup \mathcal{C}$ is dense and for every $C \in \mathcal{C}$, $f|_C$ is constant.*

PROOF

Suppose that X is a specker space and let $0 \neq f \in C(X)$. Then there is a clopen $K \subset X$ such that $f|_K$ is a nonzero constant. Let $\mathcal{C} = \{K \in \mathcal{B}(X) : f|_K \text{ is constant}\}$. If $\bigcup \mathcal{C}$ is dense, we are done. Suppose that $\bigcup \mathcal{C}$ is not dense. Then $int(X \setminus (\bigcup \mathcal{C})) \neq \emptyset$ is open. Since X is zero-dimensional, there is a $\emptyset \neq T$ clopen with $T \subset int(X \setminus (\bigcup \mathcal{C}))$. Let $g = f\chi_T$. Then $0 \neq g \in C(X)$, and g is not a non-zero constant on any clopen subset. This is a contradiction. Therefore $\bigcup \mathcal{C}$ is dense. The converse is clear.

QED

It should be pointed out that for the above result it is sufficient that X have a clopen π-base. With this observation the preceding lemma is actually a restatement of Proposition 0.5 [23]; X is a specker space if and only if it is a weakly specker space with a clopen π-base.

Suppose that $\{\mathcal{C}_i : i \in \mathbb{N}\}$ is a sequence of covers of $\mathcal{A}$. Suppose also that for each $i \in \mathbb{N}$, $\mathcal{C}_i = \{c_j : 1 \leq j \leq n_i\}$. Let $d_1 = c_1$ and, for $1 < j \leq n_i$, let $d_j = c_j \setminus \bigvee\{c_k : 1 \leq k \leq j-1\}$. Let $\mathcal{D}_i = \{d_j : 1 \leq j \leq n_i\}$. Then $\mathcal{D}_i$ is a partition of $\mathcal{A}$ and $\mathcal{D}_i \prec \mathcal{C}_i$. If $\mathcal{D}$ is a common refinement of the sequence $\{\mathcal{D}_i : i \in \mathbb{N}\}$ by a quasi-cover, then $\mathcal{D}$ is a common refinement of the sequence $\{\mathcal{C}_i : i \in \mathbb{N}\}$ by a quasi-cover. Define $\mathcal{D}'_i$ inductively by $\mathcal{D}'_0 = \mathcal{D}_0$ and $\mathcal{D}'_{n+1} = \{a \wedge b : a \in \mathcal{D}'_n \text{ and } b \in \mathcal{D}_{n+1}\}$.

Then $\{\mathcal{D}'_i : i \in \mathbb{N}\}$ is a sequence of partitions with $\mathcal{D}'_i \prec \mathcal{D}_i$ such that $\mathcal{D}'_1 \succ \mathcal{D}'_2 \succ \cdots$. If $\mathcal{D}$ is a common refinement by a quasi-cover of the sequence $\{\mathcal{D}'_i : i \in \mathbb{N}\}$, then $\mathcal{D}$ is a common refinement by a quasi-cover of the sequence $\{\mathcal{D}_i : i \in \mathbb{N}\}$. Call a sequence of covers $\{\mathcal{D}_i : i \in \mathbb{N}\}$ with $\mathcal{D}_1 \succ \mathcal{D}_2 \succ \cdots$ a *decreasing sequence of covers.* We have proved the following lemma.

Lemma 3.5.8 *A Boolean algebra $\mathcal{A}$ is specker if and only if every decreasing sequence of partitions has a common refinement by a quasi-cover.*

We are now ready to state and sketch the proof of the following theorem which is due to A. Hager.

Theorem 3.5.7 *A Boolean algebra $\mathcal{A}$ is specker if and only if its Stone dual $St(\mathcal{A})$ is a specker space.*

PROOF

Suppose $\mathcal{A}$ is is specker. Let $0 \neq f \in C(St(\mathcal{A}))$. Since $St(\mathcal{A})$ is compact and zero-dimensional, there is a finite cover of $St(\mathcal{A})$ by clopen sets such that f varies less than 2^{-n} on each set of this cover. For each n, let $\mathcal{U}_n$ be this cover. Then in $\mathcal{B}(St(\mathcal{A})) \cong \mathcal{A}$, $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is a sequence of covers. By the hypothesis, there is a common refinement by a quasi-cover $\mathcal{U}$. Then $\bigvee \mathcal{U} = cl(\bigcup \mathcal{U}) = St(\mathcal{A})$ so that $\bigcup \mathcal{U}$ is dense. Since $\mathcal{U} \prec \mathcal{U}_n$ for all n, if $K \in \mathcal{U}$ then f varies less than 2^{-n} on K for all n. Therefore is f is constant on K.

Suppose now that $St(\mathcal{A})$ is a specker space. Let $\{\mathcal{C}_i\}$ be a sequence of covers of $\mathcal{A}$. By Lemma 3.5.7, we may assume without loss of generality that $\{\mathcal{C}_i\}$ is a decreasing sequence of partitions. Via the isomorphism $\mathcal{A} \cong \mathcal{B}(St(\mathcal{A}))$ we may view these as a sequence of finite clopen partitions of $St(\mathcal{A})$. We then construct a Cantor type function $f \in C(St(\mathcal{A}))$ as the uniform limit of simple functions f_i defined on $\mathcal{C}_i$

having the property that if $C_i \in \mathcal{C}_i$ has $C_1 \supset C_2 \supset \cdots$ then there exists an $r \in \mathbb{R}$ such that $\bigcap C_i = f^{-1}(r)$ and if $r \in f^{-1}(St(\mathcal{A}))$ then $f^{-1}(r)$ is of this form. By the hypothesis and Lemma 3.5.6, there exists a $\mathcal{C} \subset St(\mathcal{A})$ with $\bigcup \mathcal{C}$ dense such that for every $C \in \mathcal{C}$, $f|C$ is constant. Viewed in $\mathcal{A}$, $\mathcal{C}$ is a quasi-cover and a common refinement of the sequence $\{\mathcal{C}_i\}$.

QED

For our purposes, we have that a compact zero-dimensional space X is a specker space if and only if $\mathcal{B}(X)$ is a specker boolean algebra. This is where we can make a translation of our original problem. All of the results used in this paragraph can be found in [27] Chapters 3 and 6. Suppose that X is a compact zero-dimensional specker space with the countable chain condition. Then $\mathcal{B}(X)$ is a specker boolean algebra with the countable chain condition. The irreducible map $e : EX \to X$ induces an o-large embedding of $\mathcal{B}(X)$ in $\mathcal{B}(EX)$, and a boolean isomorphism of $\mathcal{R}(X)$ and $\mathcal{R}(EX)$. Since EX is extremally disconnected, $\mathcal{R}(EX) = \mathcal{B}(EX)$. Identifying isomorphic algebras, we have an o-large embedding of $\mathcal{B}(X)$ in $\mathcal{R}(X)$. Since $\mathcal{R}(X)$ is the completion of $\mathcal{B}(X)$, our original question translates to the following. Is the completion of a specker boolean algebra with the countable chain condition a specker boolean algebra? What follows is part of an attempt to answer this question.

Definition 3.5.6 *Let $\{\mathcal{C}_i : i \in \mathbb{N}\}$ with $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$ be a decreasing sequence of covers of $\mathcal{A}$. $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$ is said to be a binary sequence of covers if for every $i \in \mathbb{N}$ and for every $c \in \mathcal{C}_i$, $|\{b \in \mathcal{C}_{i+1} : b \leq c\}| \leq 2$.*

We will show that specker boolean algebras can be characterized in terms of binary sequences of covers. We first need some terminology and notation. For a boolean algebra $\mathcal{A}$, $C \subset \mathcal{A}$ is called a *chain* if C is a linearly ordered subset of $\mathcal{A}$. For a decreasing sequence of covers $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$, a chain $c_1 \geq c_2 \geq \cdots$ is a *representative*

chain if $c_i \in \mathcal{C}_i$ for all $i \in \mathbb{N}$. A chain $c_1 \geq c_2 \geq \cdots$ is called a *proper chain* if there exists an $i \neq j$ such that $c_i \neq c_j$, provided that $\{\mathcal{C}_i\}$ is not a trivial sequence. For $a \in \mathcal{A}$, let $[0, a] = \{b \in \mathcal{A} : 0 \leq b \leq a\}$. For $r \in \mathbb{R}$, let $\lfloor r \rfloor$ denote the largest integer less than or equal to r and let $\lceil r \rceil$ denote the least integer greater than or equal to r.

We first need the following technical lemma.

<u>Lemma 3.5.9</u> *Let $\mathcal{A}$ be a boolean algebra. If for every binary sequence of covers $\mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots$ there is a proper representative chain $\{b_i\}$ with a nonzero lower bound, then for every sequence of covers $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$ there is a proper representative chain $\{c_i\}$ with a nonzero lower bound .*

PROOF

By way of contradiction, suppose that $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$ is a sequence of covers such that every proper representative chain $\{c_i\}$ has $\bigwedge\{c_i : i \in \mathbb{N}\} = 0$. We will construct a binary sequence of covers with the same property.

The idea of this proof is to construct, for every $\mathcal{C}_i \succ \mathcal{C}_{i+1}$, a binary sequence of covers " between" $\mathcal{C}_i$ and $\mathcal{C}_{i+1}$. We do this pointwise for each element a of $\mathcal{C}_i$ by taking pairwise joins of the elements of $\mathcal{C}_{i+1}$ that are below a.

Let $i \in \mathbb{N}$ and for $a \in \mathcal{C}_i$, let $\mathcal{C}_{i+1}(a) = \{c \in \mathcal{C}_{i+1} : c \leq a\}$. Then $\mathcal{C}_{i+1}(a)$ is a binary sequence of covers of $[0, a]$. Suppose that we have constructed $\mathcal{B}'_k(a) \succ \mathcal{B}'_{k-1}(a) \succ \cdots \succ \mathcal{B}'_1(a) = \mathcal{C}_{i+1}(a)$ a binary sequence of covers of $[0, a]$. Construct $\mathcal{B}'_{k+1}$ as follows. First there is an indexing of the elements of $\mathcal{B}'_k(a) = \{b_n : 1 \leq n \leq m_k\}$. For $1 \leq n \leq \lfloor \frac{m_k}{2} \rfloor$, let $a_n = b_{2n-1} \vee b_{2n}$ and if m_k is odd, let $a_{\lceil \frac{m_k}{2} \rceil} = b_{m_k}$. Now take $\mathcal{B}'_{k+1} = \{a_n : 1 \leq n \leq \lceil \frac{m_k}{2} \rceil\}$. For some finite $k \in \mathbb{N}$, $\mathcal{B}'_k = \{a\}$. Let $\mathcal{B}_n = \mathcal{B}'_{k-n}$ for $1 \leq n \leq k$. We have constructed $\{a\} = \mathcal{B}_1(a) \succ \mathcal{B}_2(a) \succ \cdots \succ \mathcal{B}_k(a) = \mathcal{C}_{i+1}(a)$ a binary sequence of covers of $[0, a]$. Now repeat this process for each $a \in \mathcal{C}_i$. Suppose that $\mathcal{C}_i = \{a_n : 1 \leq m \leq m_i\}$. Let k_n be such that $\mathcal{B}_{k_n}(a_n) = \mathcal{C}_{i+1}(a_n)$ for $1 \leq m \leq$

m_i. Let $k = max\{k_n : 1 \leq n \leq m_i\}$. We need to do some manipulation of the indices. For each $1 \leq n \leq m_i$ and for each $j \in \mathbb{N}$ with $k_n \leq j \leq k$, let $\mathcal{B}_j(a_n) = \mathcal{B}_{k_n}(a_n)$. Now for $1 \leq l \leq k$, let $\mathcal{B}_l = \mathcal{B}_l(a_1) \cup \mathcal{B}_2(a_1) \cup \cdots \cup \mathcal{B}_l(a_{m_i})$. We have constructed $\mathcal{C}_i = \mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots \succ \mathcal{B}_k = \mathcal{C}_{i+1}$ a binary sequence of covers of $\mathcal{A}$. Repeat this process for every pair $\mathcal{C}_i \succ \mathcal{C}_{i+1}$ of the original sequence of covers and reindex the resulting binary sequence so that $\mathcal{C}_1 = \mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots$. Then for each $\mathcal{C}_i$ there is an i_j such that $\mathcal{C}_i = \mathcal{B}_{i_j}$. Now let $\{a_n\}$ be a proper representative chain of the sequence $\mathcal{B}_n$. By construction there is a chain $\{a_{i_j}\}$ with $\{a_{i_j}\} \in \mathcal{C}_i$ which is cofinal in $\{a_n\}$. Since $\{a_n\}$ is a proper chain and $\{a_{i_j}\}$ is cofinal, $\{a_{i_j}\}$ is a proper representative chain of the sequence $\{\mathcal{C}_i\}$. By the hypothesis, $\bigwedge\{a_{i_j}\} = 0$. As any lower bound for $\{a_n\}$ is a lower bound for $\{a_{i_j}\}$, we have that $\bigwedge\{a_n\} = 0$

QED

We then have the following theorem which allows us to characterize specker boolean algebras in terms of binary sequences of covers.

<u>Theorem 3.5.8</u> *Let $\mathcal{A}$ be a boolean algebra. $\mathcal{A}$ is specker if and only if for every binary sequence of partitions $\mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots$ there is a proper representative chain $\{b_i\}$ such that $\{b_i\}$ has a nonzero lower bound.*

PROOF

Suppose that $\mathcal{A}$ is specker. Let $\mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots$ be a binary sequence of partitions. Let $\mathcal{B}$ be a common refinement by a quasi-cover. Since $\bigvee \mathcal{B} = 1$, there is a $0 \neq b \in \mathcal{B}$. Since $\mathcal{B}$ is a common refinement, for each $\mathcal{B}_i$ there is a $b_i \in \mathcal{B}_i$ with $b \leq b_i$. Since each $\mathcal{B}_i$ is a partition, $\{b_i\}$ is a representative chain. If $\mathcal{B}_1 \succ \mathcal{B}_2 \succ \cdots$ has no representative chains $\{b_i\}$ with $b_i \neq b_j$ for some $i \neq j$, then we can take $\mathcal{B} = \mathcal{B}_1$ and the sequence is trivial.

Suppose that $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$ is a decreasing sequence of partitions of $\mathcal{A}$. By Lemma 3.5.6, it suffices to show that this sequence has a common refinement by a quasi-cover. By the hypothesis and Lemma 3.5.7, there is a proper representatice chain $\{c_i : i \in \mathbb{N}\}$ such that $\{c_i : i \in \mathbb{N}\}$ has a nonzero lower bound. Let L be the set of all nonzero lower bounds for all proper representative chains of $\mathcal{C}_1 \succ \mathcal{C}_2 \succ \cdots$. Then L is a common refinement of the $\mathcal{C}_i$'s. It remains to show that L is a quasi-cover. If $\bigvee L = 1$ we are done. Suppose that $\bigvee L \neq 1$. Then there is an $a \in \mathcal{A}$ with $0 < a < 1$ such that a is an upper bound for L. Let $b = a'$ and let $\mathcal{C}_i[b] = \{c \wedge b : b \in \mathcal{C}_i\}$. Then $\mathcal{C}_i[b]$ is a decreasing sequence of partitions of $[0, b]$. Now, $\mathcal{C}_1[b] \cup \{a\} \succ \mathcal{C}_2[b] \cup \{a\} \succ \cdots$ is a decreasing sequence of partitions of $\mathcal{A}$. Therefore, by the hypothesis and Lemma 3.5.9, there is a proper representative chain $\{b_i\}$ with a nonzero lower bound. Since $a \wedge b = 0$, $b_i \in \mathcal{C}_i[b]$ for all i. For each i, $b_i = c_i \wedge b$ for some $c_i \in \mathcal{C}_i$. Since $\{b_i\}$ is a chain and each $\mathcal{C}_i$ is a partition, $\{c_i\}$ is a chain. Since $\{b_i\}$ is a proper chain, so is $\{c_i\}$. If d is a nonzero lower bound for $\{b_i\}$ then it is also a lower bound for $\{c_i\}$ and therefore $d \leq a$. This is a contradiction since $d \leq b$ and $a \wedge b = 0$. Therefore, $\bigvee L = 1$ and the proof is complete.

QED

The current goal with these results is to show that if X is a compact specker space with the countable chain condition, then X contains a dense set of isolated points. We begin by assuming that X has no isolated points. Then $\mathcal{B}(\mathcal{X})$ is an atomless specker boolean algebra. It may be possible to then construct an uncountable anti-chain, violating the countable chain condition. This will show that X contains one isolated point. It should then be possible to argue that X in fact contains a dense set of isolated points, and therefore that EX is a specker space.

CHAPTER 4
CONCLUSION

The focus of this dissertation is an examination of f-rings which are rich in idempotents. We consider three different types of f-rings, reflecting three different degrees of "richness".

In Chapter 2, we consider local-global f-rings. For bounded rings, we obtain the equivalence of the following conditions.

1. A is a local-global ring.

2. Every primitive quadratic polynomial with non-negative coefficients in A represents a multiplicative unit.

3. $Max(A)$, the maximal spectrum, is zero-dimensional.

As one measure of the richness of idempotents for local-global f-rings, recall that in the verification of the local-global condition, we began with a primitive polynomial $f \in A[t]$ and constructed an element $s \in A$ such that $f(s)$ is a multiplicative unit. This element s we constructed is, in fact, a linear combination of idempotents. The zero-dimensionality of $Max(A)$ also gives another indication of the richness of idempotents in local-global rings. Here we need to recall that the clopen subsets of $Max(A)$ are the basic open sets determined by an idempotent of A and that the zero-dimensionality of $Max(A)$ implies that the clopen sets are a base for the open sets of $Max(A)$. Considering the case $A = C(X)$ for a Tychonoff space X, we obtain a nice addition to the "Algebra-Topology Dictionary". Namely, that a space X is strongly zero-dimensional if and only if every primitive quadratic polynomial with

non-negative coefficients in $C(X)$ represents a unit. This characterization is of additional interest because it provides a first order algebraic characterization of strongly zero-dimensional spaces. For this reason, this condition merits further investigation. Chapter 2 also left a glaring open problem. That is, whether or not all of the above conditions are equivalent in the unbounded case. As indicated at the end of the chapter, the problem is one of "cutting down" primitive polynomials to the bounded subring $A(1)$ of A and maintaining primitivity. We give sufficient conditions for this to occur, namely, that the ring be a Bezout ring. The working conjecture is that the general result does hold, but the proof may require some sort of set theoretical forcing argument.

In Chapter 3, we consider two related measures of the richness of idempotents. Recall that an f-ring A is said to be of specker-type if A is an essential extension of $S(A)$, the subalgebra generated by the idempotents of A, and A is said to be a quasi-specker ring if A is an o-essential extension of $S(A)$. Recall also that for a ring A, $Q(A)$ denotes the complete ring of quotients of A. We first obtain the equivalence of the following conditions.

1. A is a specker-type ring.

2. A is an f-subring of $Q(S(A))$.

3. $S(A)^L = Q(A)$.

In particular, condition (3) gives some indication of the richness of idempotents in specker-type f-rings. It says, in effect, that every element of the complete ring of quotients of A can be written as a (possibly infinite) linear combination of idempotents. For quasi-specker f-rings, the richness of idempotents is indicated by the defining condition that every nonzero element is larger than some scalar multiple of a

nonzero idempotent. Results regarding specker-type and quasi-specker rings become particularly interesting for the case $A = C(X)$. Recall that a space X is said to be a specker space (quasi-specker space) if $C(X)$ is a specker-type (quasi-specker) ring. For specker spaces, we have in addition to the previous conditions, suitably translated, the following.

4. For every nonzero $f \in C(X)$, there is a clopen set on which f is nonzero and constant.

In the case that X is compact we are able to improve this characterization to,

5. For every $f \in C(X)$ there is a collection of clopen sets whose union is dense in X and such that f is constant on each of these clopen sets.

Again, confusing the clopen sets of X with the idempotents of $C(X)$, we get some sense of the richness of idempotents for specker spaces. For compact quasi-specker spaces, the richness of idempotents of $C(X)$ is reflected in the condition which characterizes these spaces as those which have a countable clopen π-base.

Motivated by the "tight" containment of $C(X)$ in $Q(X)$, we consider the specker condition for spaces vis-a-vis their absolutes. We are able to show that if a space X is a quasi-specker space and EX, the absolute of X, is a specker space, then X is a specker space. As a partial converse, we have the conjecture that if X is compact with the countable chain condition and X is a specker, then EX is a specker space. As a special case of this conjecture we are able to show that if X compact, with a countable π-base, then X is a specker space implies that EX is a specker space. Finally, via A. Hager's result on specker boolean algebras and specker spaces, we are able to translate the problem of compact specker spaces to one of specker boolean algebras. Here, using the characterization obtained for specker algebras in terms of

refinements of binary sequences of covers, we hope to prove the following. If X is a compact specker space with the countable chain condition, then X contains a dense set of isolated points. Then EX will also contain a dense set of isolated points and therefore be a specker space.

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BIOGRAPHICAL SKETCH

Scott David Woodward was born in Phoenix, Arizona, in 1955. Before returning to school in 1980, he was a journeyman mason working out of Orlando, Florida. He received his B.S. and M.S. degrees in mathematics from the University of Florida in 1983 and 1987 respectively. He is married to Rita Wendt-Woodward and has two children, Christopher and Michael.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Jorge Martinez, Chairman
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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